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Nonlocal elasticity in plates using novel trial functions

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ABSTRACT

This study presents the Ritz formulation, which is based on boundary characteristic orthogonal polynomials (BCOPs), for the two-phase integro-differential form of the Eringen's nonlocal elasticity model. This approach is named the nonlocal Ritz method (NL-RM). This feature greatly reduces the computational cost compared to the nonlocal finite-element method (NL-FEM). Another advantage of this approach is that, unlike NL-FEM, the nonlocal mass and stiffness matrices are independent of the mesh distribution. Here, these formulations are applied to study the static-bending and free-dynamic analyses of the Kirchhoff plate model. In this paper, novel 2D BCOPs of the plate are derived as coordinate functions. These polynomials are generated using a modified Gram-Schmidt process and satisfy the given geometrical boundary conditions as well as the natural boundary conditions. The accuracy and convergence of the presented model, demonstrated through several numerical examples, are discussed. A concise argument on the advantages of NL-RM compared to NL-FEM is also provided.

1. Introduction

The classical (local) continuum theories assume that the strain and stress at each point are related. However, these theories have been shown to be inadequate for numerous situations in which a characteristic length scale of the medium must be considered in the physical solution. The local theory cannot be used to describe the stress and strain fields around sharp crack tips, the dispersion of elastic waves, strain softening, size-dependent effects and dislocation [[3]]. As a result, nonlocal continuum theories are needed to model the structural responses of new materials to account for small-scale effects. Nonlocal theories assume that the stress at each point is affected by the strain at all points in the field. Kröner [30], Kunin [32], and Krumhansl [31] proposed for the first time the idea of the nonlocal theory. Among size-dependent theories, one of the most well-known is the nonlocal continuum theory of Eringen. In this theory, the scale effect and long-range interatomic interactions are entered as material parameters into the constitutive equations. Later, Edelen and Laws [13], Edelen et al. [12], and Eringen and Edelen [17] improved nonlocal formulations in a thermodynamic framework and accounted for long-range interactions in the constitutive equations in an integral form. Eringen [16] and Altan [2] presented the two-phase integro-differential nonlocal elasticity theory, which includes both local and nonlocal integral-type elasticity theories by assigning a volume fraction to each of the theories. In integral non-local theory, an integral operator is represented as a material parameter to take into account the nonlocal nature of the material structure. In this theory, the stress at a material point is dependent on a positive distance-decaying kernel function as a weighted integral of strains over a specified finite region. This theory for isotropic material results in a set of integro-partial differential equations for the displacement domain, which are difficult to solve, particularly for mixed boundary-value problems [40].

The integral nonlocal elasticity theory was revised by Polizzotto [38], who proposed nonlocal finite-element models to remove the difficulties of employing the nonlocal boundary conditions. Polizzotto [38] obtained nonlocal finite element (NL-FEM) and an alternative FEM-based iterative formulation of the integral-type nonlocal model based on three variational principles. A nonlocal-type FEM (NL-FEM) was developed in which the symmetric global-stiffness matrix includes the nonlocal characteristics of the problem. Moreover, an iterative-FE-based solution method (Iterative-FEM) was presented in which the local strain energy is iteratively corrected by an imposed correction strain. The NL-FEM may be used to solve one- and two-dimensional nonlocal elastic problems. Pisano and Fuschi [35] investigated an elastic bar subjected to tension based on Eringen's nonlocal integral-type model by transforming the governing equation into the standard solvable Volterra integral equation of the second type. Later, Benvenuti and Simone [5], proposed a closed-form solution of the local-nonlocal strain-stress law for a homogeneous rod subjected to different load cases by reducing the integro-differential boundary value problem to a differential one. An NL-FEM was developed, in detail, to solve 2D elastic problems (in-plane motion) for homogeneous [36] and non-homogeneous [37] materials based on the two-phase integro-differential model.

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Furthermore, the integral nonlocal elasticity was reduced to differential nonlocal elasticity for certain special kernel functions by Eringen [15]. The differential nonlocal elasticity leads to a set of singular differential equations, and these equations could be simply solved; however, some difficulty exists when employing natural boundary conditions [40]. In the literature, the nonlocal differential model has been most widely used for bending, buckling and vibration analysis of nanorods, nanoplates and nanobeams [1,11,22,33,41]. Also this nonlocal model is used for studying the vibration and buckling of functionally graded rectangular nano-plates based on nonlocal exponential shear deformation [27,29]. Some authors addressed the well-known paradoxical cantilever nonlocal beam problem [9,10,42], where an unreasonable stiffening effect was found in their results. Challamel and Wang [9] also noted that this paradox can be solved with an integral-based model that combines the local and nonlocal curvatures in the constitutive relations. Most recently, Khodabakhshi and Reddy [26] proposed a general finite-element formulation for the two-phase integro-differential nonlocal model to solve the well-known paradoxical cantilever nonlocal beam.

In all of the references outlined above, FEM-based approaches (i.e., classical and nonlocal-type FEM) have been used as the solution methods. Shaat [40] stated that applying FE-based approaches to analyse nonlocal integral-type elastic problems required extremely challenging computational efforts. Furthermore, Khodabakhshi and Reddy [26] noted that in the FE discretized integral model, due to the existence of the non-zero terms in the global stiffness matrix, the properties of the mesh distribution and also the need to increase the mesh size to achieve the desired accuracy, this approach demands a high computational cost.

The main contribution of this paper is the presentation of an efficient computational method to overcome these obstacles. To achieve this goal, the Ritz formulation based on the boundary characteristic orthogonal polynomials (BCOPs) for the two-phase integro-differential nonlocal elasticity model is presented. In addition, a novel set of BCOPs are derived based on the approach described by Bhat [6] as trial functions in the Ritz method. These polynomials are generated using a modified Gram-Schmidt orthonormalization process. The advantage of the novel BCOPs is that not only the given geometrical boundary conditions are satisfied but also the natural boundary conditions. It should be noted that this method considerably improves the problem related to the bandwidth growth of the stiffness matrix in FEM-based approaches [26] because the orthogonality property of the BCOPs would lead to an increased number of zero entries in the stiffness matrix.

The nonlocal differential model has been used with the Rayleigh–Ritz method to calculate the natural frequencies of uniform and non-uniform nonlocal plates for several possible boundary conditions [4,8]. Rayleigh–Ritz method has been used for local vibration analysis of moderately thick rectangular plates [24] and functionally graded rectangular plate [28]. Faroughi and Goushegir [18] studied the in-plane natural frequencies and mode shapes of non-uniform rectangular nanoplates using the Eringen's nonlocal differential model along with the Rayleigh–Ritz method. Regarding the Eringen's nonlocal differential model with the Rayleigh–Ritz method for modeling a nanoplate, to the best knowledge of the authors, the Ritz method using novel BCOPs has not been used to model two-phase integro-differential nonlocal elasticity. This efficient computational method is implemented here to study the static and dynamic analyses of two-phase integro-differential nonlocal plate problems. To date, this has not been carried out using any numerical approaches. It is noteworthy that the method implemented here completely eliminates the challenges of generating elements within the influence zones (i.e., cohesive zones) in the NL-FEM.

The outline of the paper is as follows. Section 2 describes two-phase integro-differential nonlocal theory in 2D. The kernel function is expressed in Section 3. Section 4 explains the Ritz method for nonlocal Kirchhoff plate theory. The construction of 2D novel BCOPs is explained in Section 5. Numerical examples are given in Section 6. Finally, some conclusions are drawn in Section 7.

2. Two-phase integro-differential nonlocal theory in two dimensions

Eringen's nonlocal theory [14] assumes that the stress at a reference point $\mathbf{x}$ in the body is dependent not only on the strain at $\mathbf{x}$ but also on the strain field at all other points ($\mathbf{x'}$) of the material. In the general integral-type nonlocal theory, this dependency is expressed as a weighted convolution integral in which the weighting function is a scalar kernel function $\mathrm{H}(\mathbf{x}, \mathbf{x'}, l_c)$. In Eringen's integral-type nonlocal theory, the stress at the point $\mathbf{x} \in \mathrm{V'}$ is given as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_{\mathrm{V'}} \mathrm{H}\left(\mathbf{x},\ \mathbf{x'},\ l_c\right)\mathbf{D} : \epsilon\left(\mathbf{x'}\right)\mathrm{dV'} \quad (1)$$

where $\epsilon(\mathbf{x'})$, $\mathbf{D}$ and $\mathrm{V'}$ denote the local strain at $\mathbf{x'}$, the fourth-order tensor of classical linear elastic-material moduli and the nonlocal continuum volume, respectively. The parameter l_c is the length-scale parameter.

According to Eringen [16] and Altan [2], both the local and nonlocal elastic models can be combined linearly and expressed as a more general two-phase nonlocal model. Eq. (1) can be modified for a two-phase model to give

$$\boldsymbol{\sigma}(\mathbf{x}) = \eta_1\mathbf{D} : \epsilon(\mathbf{x}) + \eta_2 \int_{\mathrm{V'}} \mathrm{H}\left(\mathbf{x},\ \mathbf{x'},\ l_c\right)\mathbf{D} : \epsilon\left(\mathbf{x'}\right)\mathrm{dV'} \quad (2)$$

Where, volume fractions η_1 and η_2 denote local and nonlocal phases of the body material, respectively. The local η_1 and nonlocal-phase parameters η_2 are positive constants that should satisfy the following relation.

$$\eta_1 + \eta_2 = 1 \quad (3)$$

This model introduces two independent variables: the length scale, l_c, and local-phase parameter, η_1. However, the differential and integral forms of the model of Eringen each considered only one length scale, l_c. l_c depends on the internal characteristic length l_{in} by $l_c = e_0 l_{in}$, where, e_0 is a non-dimensional small length scale coefficient (lattice parameter, granular size or molecular diameters) and is appropriate with the description of the each nanostructure material that has to be calibrated with respect to dispersive wave properties of the Born–Kármán dynamics [15], phonon dispersion curves [21], atomistic models or reliable experimental measurements.

3. Kernel function

The kernel function $\mathrm{H}(\mathbf{x}, \mathbf{x'}, l_c)$ imposes the shape of the nonlocal influence limited to a certain radial distance induced at $\mathbf{x}$ by the strain field at the points $\mathbf{x'}$ all over the body.

The kernel function $\mathrm{H}(\mathbf{x}, \mathbf{x'}, l_c)$ has the following features [15]:

* It is a positive function that has its maximum at the point $\mathbf{x} = \mathbf{x'}$ and is attenuated by increasing $\|\mathbf{x} - \mathbf{x'}\|$.
* It reverts to a delta function $\delta(\mathbf{x}, \mathbf{x'})$ as l_c approaches zero (i.e., when l_c is negligible, the constitutive equations simplify to the classical local equations.).
* It satisfies the normalization condition (where $\mathrm{V'}$ is embedded in an indefinite domain $\mathrm{V'}_\infty$):

$$\int_{\mathrm{V'}_\infty} \mathrm{H}\left(\mathbf{x}, \mathbf{x'}, l_c\right)\mathrm{dV'} = 1 \quad (4)$$

* It is a bi-symmetric function:

$$\mathrm{H}\left(\mathbf{x}, \mathbf{x'}, l_c\right) = \mathrm{H}\left(\mathbf{x'}, \mathbf{x}, l_c\right) \quad (5)$$

* It approximates atomic lattice theory when l_c approaches the external characteristic length.

In the present study, the kernel function is chosen based on a modified non-singular stress-gradient kernel function proposed by Ghosh et al. [20]. The 2D stress-gradient kernel is defined as