International Journal of Mechanical Sciences 123 (2017) 64–73

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Accurate Eigenvector-based generation and computational insights of Mindlin's plate modeshapes for twin frequencies

Nabanita Datta*, Yogesh Verma

Indian Institute of Technology, Kharagpur, India

A R T I C L E I N F O

Keywords:
Plate vibration
Modeshapes
Eigenvectors
Nodal patterns
Trial functions

A B S T R A C T

A semi-analytical approach to understand the manifestation of plate modeshapes associated with twin frequencies has been presented. Square Mindlin's plate, clamped on all sides, has been considered here. It highlights the importance of efficacy of the beam-wise trial functions in an energy-based plate vibration analysis method, in terms of (a) accuracy, (b) orthogonality, (c) sense (plus/minus) and (d) interference. The inconsistency in the modeshapes of repeated frequencies, seen extensively in literature, has been attempted to be removed, through superior closed-form orthogonal set of Timoshenko admissible functions into the Rayleigh-Ritz method. The constructive/destructive interferences of the admissible functions, which are the products of the beam-wise modeshapes, give the final nodal patterns and the prominence of the anti-nodes. Also, the pairs of 'very close' but distinct frequencies, which were often considered as 'numerical errors', have been counter-intuitively justified through their Eigenvectors, which are either symmetric or skew-symmetric in the matrix form. Nodal patterns for CCCC plate modeshapes are accurately investigated; i.e. chess-board and diagonal nodal patterns.

1. Introduction

Very little literature is available in the area of free vibration of plates which mathematically study plate modeshapes through the Eigenvectors of the Eigenvalue problem. Numerical and experimental attempts have been made to visualize the plate modeshapes, with limited success, in the first few frequencies only. Experimentally, modeshapes of the duplicate frequencies are not distinguishable, because they cannot co-exist simultaneously at a particular resonant excitation frequency of the vibration shaker. In real life, creating the exact classical clamped boundary condition can be a huge challenge, which leads to the invocation of random frequencies corresponding to plates with other undesired classical/non-classical boundary conditions. The premise of this work is as follows :

- What causes inaccuracies in plate modeshapes generated by numerical techniques?
- Why are plate modeshape often unpredictable in published literature?
- Why do the duplicate frequencies have dissimilar modeshapes by some numerical methods, while 'very close' pair of frequencies have unpredictable modeshapes in some other numerical methods?
- Why do duplicate frequencies produce different plate modeshapes by numerical methods? This appears as a mathematical discrepancy. Repeated roots of an Eigenvalue problem have different Eigenvectors, but are they supposed to manifest different or duplicate modeshapes?
- How important is the efficacy of the beam-wise trial functions in an energy-based plate vibration analysis method, in terms of (a) accuracy, (b) orthogonality, (c) sense (±), (d) interference?
- Only the frequency magnitude is no indication of the energy spread across the area of the plate. Frequency is the Eigenvalue, i.e. just a number, which does not indicate the spatial configuration of the vibration, which is important in dynamic stress analysis in various engineering applications.

2. Literature review and Work overview

Ma and Huang [15] studied the modeshapes of a plate clamped on all sides (CCCC plate), and reported the first 12 frequencies, compared with FEA-based results. The experimental modeshape of the duplicate frequencies (Mode# 6) does not match the numerical ones. Mode# 11 shows a chess-board configuration, while the experiment shows a shaky diagonal nodal line. This questions the efficacy of either method, and necessitates the establishment of a robust approach to generate the accurate plate modeshapes.

* Corresponding author.
E-mail address: nabanitadatta@gmail.com (N. Datta).

http://dx.doi.org/10.1016/j.ijmecsci.2017.01.044
Received 22 September 2016; Received in revised form 20 January 2017; Accepted 26 January 2017
Available online 29 January 2017
0020-7403/

Nomenclature

a, b, h	Length, breadth and thickness of the plate (m,m,m)
A	Cross-sectional area of Timoshenko beam (m^2)
A_S	Aspect ratio of plate b/a (dimensionless)
c_{ij}, d_{ij}, e_{ij}	Weights of the assumed Rayleigh-Ritz assumption
$C_j, \overline{C}_j$	Arbitrary constants of Timoshenko beam and pure bending slope modeshapes
D	Plate rigidity (Nm)
E, G	Young modulus, Shear modulus (N/m^2)
I	Area moment of inertia of cross section (m^4)
k^2	Shear correction factor for shear strain (dimensionless)
K, M	Stiffness matrix and mass matrix
L	Length of the beam (m)
x, y, t	Independent variable in length, breadth and time (m, m, s)
U, T	Strain energy and kinetic energy (Joule)
$w(x, y; t)$	Lateral displacement of the plate (m)
$W(x, y)$	Plate modeshape (m)
W_{xi}, W_{yj}	Beam modeshape in x and y-direction (m)
ρ, ν	Density of the material, Poisson ratio (kg/m^3)
ξ, η	Non-dimensional length, breadth of the beam/plate (dimensionless)
ω, Ω	Non-dimensional and dimensional natural frequencies (dimensionless, rad/s)
$\Psi_x(\xi, \eta), \Psi_y(\xi, \eta)$	Pure bending slope of plate in x-direction, y-direction (dimensionless)
$\Psi_{xi}, \Psi_{xj}; \Psi_{yi}, \Psi_{yj}$	Pure bending slope modeshape of beam in x-direction, y-direction (dimensionless)

Chen et al. [3] studied thin plates both experimentally and numerically, and reported the first 15 frequencies and their modeshapes. For the 6 (six) pairs of repeated frequencies (one index is even and the other is odd), the nodal lines are sometimes chess-board, sometimes diagonal, sometimes shaky-diagonal. Also, there are discrepancies between the experimental and numerical modeshapes. The pairs of 'very close' frequencies have not been reported in totality: one of the two in each pair is missing consistently. These frequencies are very close to each other but the modeshapes are very different. The resonant excitation of the plate by the exciter often misses out on one of the two frequencies. This necessitates the mathematical investigation of modeshapes and frequencies, in order to acknowledge all of the resonant frequencies in the range of experimental excitation. Ferreira et al. [7] showed the first $2 \times 2 = 4$ modes of CCCC plates, insufficient for the first pair of 'very close' frequencies to appear. Also, the duplicate frequencies do not show exactly mirror-image modeshapes, in spite of having the same Eigenvalue (frequency). Ferreira et al. [8] shows the first 8(eight) vibration modeshapes of a square CCCC isotropic plate, obtained through the numerical analysis of first-order and higher-order shear deformation theories. The first pair of duplicate frequencies has mirror-image modeshapes, but their Eigen frequencies are not exactly equal. The next pair, supposed to be equal, has different-looking modeshapes, and the Eigenvalues are also not exactly equal. Bardell [2] used the Kirchhoff's plate theory, giving the first $2 \times 2 = 4$ modes of CCCC plates, with the duplicate frequencies showing chess-board modeshapes. Shojaee et al. [21] used NURBS for Kirchhoff thin plates, and showed the first 20 (twenty) modeshapes of a square CCCC plate. Here again, the duplicate repeated frequencies suffer from shaky nodal lines (especially 9th mode), and uncertain hazy "chess-board like" nodal patterns. Zhang et al. [24] showed the first 8 modeshapes of a CCCC graphene sheet using the classical plate theory, involving cubic splines as shape functions. The repeated frequencies again showed a chessboard-like configuration, while an ANSYS analysis of the square CCCC plate clearly gives exactly diagonal nodal lines with mirror-images for the pair of repeated frequencies. Also, their work was unable to generate any of the pair of 'very close' frequencies.

Saha et al. [18] presented the modeshapes of a Mindlin's plates with all edges clamped (CCCC plate), among a few other boundary conditions. However, this was limited to only the first $2 \times 2 = 4$ modeshapes. Discrepancies were consistently visible for the plate modeshapes with the duplicate frequencies. Liew et al. [14] showed the first 12 modeshapes of a square CCCC Mindlin's plate. The duplicate frequencies show a chess-board configuration, which is contrary to those shown by FEA Ferreira [7] and experiments (Ma and Huang [15]). Aridogan et al. [1] analytically modelled a thin CCCC plate, and presented only the first $2 \times 2 = 4$ modes. Hou et al. [9] used the DSC-Ritz method to study the plate with thickness ratio 0.1, and showed the first six frequencies and modeshapes, with some inaccuracies in the nodal lines, especially at the edges and/or corners.

The doubts, discrepancies, and deductions from the above literature review (Table 1) are as follows :

1) Most published works show plate frequencies as a list in the ascending order of magnitude, without distinguishing them on the basis of their Eigenvectors/modeshapes.
2) The duplicate frequencies cannot be distinguished experimentally. At a single exciter frequency, one modeshape will be manifested and the other, suppressed. The duplicate frequencies are numerically/semi-analytically recognized, but some pairs of frequencies which are very close to each other, are often mistaken to be duplicates, too. Some researchers assume 'numerical errors' in not being able to achieve certain "duplicate" set of frequencies.
3) The nodal lines of the duplicate frequencies should be chess-board or diagonal or shaky-diagonal? There are conflicting conclusions from experimental, numerical, and semi-analytical methods (Table 1). Intuitively, the identical Eigenvalues should generate mirror-image modeshapes. Chess-board configuration indicates that the Eigenvector has only one dominant element, i.e. only one

Table 1
Comparison of modeshapes in literature, for various types of frequencies.

	Trial function	Unique freq.	Duplicate frequencies	'Very close' freq.
Bardell [2]	Legendre polynomials	✓	Chess-board	Not studied
Saha et al. [18]	Dynamic Timoshenko	✓	chessboard, diagonal	Not studied
Ma, Huang [15]	–	✓	Shaky	✓
Liew et al. [14]	Cubic Spline	✓	Chess-board	✓
Chen et al. [3]	–	✓	Shaky	✓
Ferreira [7]	Cubic polynomial	✓	shaky, diagonal	Not studied
Ferreira et al. [8]	Legendre polynomials	✓	Shaky	✓
Shojaee et al. [21]	NURBS	✓	Chess-board	✓
Aridogan et al. [1]	Euler-Bernoulli	✓	Chess-board	Not studied
Zhang et al. [24]	Cubic Spline	✓	Chess-board	Not manifested
Our Work	Dynamic Timoshenko	✓	Diagonal (h/a = 0.01), chess-board (h/a = 0.1,0.2)	✓