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Composite expansion-Ritz method to post buckling problem of thin cold rolled strip

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ABSTRACT

This article focuses on evaluating the large deflection of the thin strip under practical condition of residual stress after cold rolling. In order to describe the problem mathematically, the incompatible von Kármán equations were introduced as the governing equations. Given the deflection of the strip along the rolling direction presenting the periodic form, the incompatible von Kármán equations along with the free boundary conditions were simplified to be a nonlinear boundary value problem in dimensionless form and turned out to be a boundary layer problem. Then composite expansion-Ritz method (CERM) was proposed to solve the problem. The composite expansion method was used to determine the form of deflection, while the geometry parameters, such as wave length, positions of boundary layers, were obtained by Ritz method. Finally, the accuracy of CERM is verified by actual measured data.

1. Introduction

Buckling induced by the residual stress is a common phenomenon in strip cold rolling and is usually called flatness defect. Due to the fact that a free thin strip bends in harmonic form to reduce the total potential energy within compressive stress region, waviness in edge and center tend to be the most typical flatness defect patterns. During the rolling process, the strip maintains flat under the rolling tension. However, in subsequent annealing or hot galvanizing processes, the latent defect as a result of residual stress (normal component along rolling direction) of strip would be converted to the manifested defect (large deflection), since the tension in these processes is quite small. Coman [7,8] clearly pointed out the incompatibility nature of the rolled strip buckling and estimated the critical load and the wavelength of the buckling waves with piecewise linear distribution of residual stress. Therefore, an essential description of deflection in terms of residual stress is of significance to get deeper insights into the post buckling of cold thin strip.

Large elastic deformations of thin plates/strips with incompatible strains or continuously distributed defects have been well studied. A modification on von Kármán equation for flexible elastic plates with dislocations and disclination was done by Zubov [26]. Considering both isolated and continuously distributed defects, several solutions of membrane containing distributed disclinations were discovered with the assumption of complete internal stress relaxation. Lewicka et al. [15] introduced the concept of 'Growth' representative of the initial influences of the process on thin strips, such as plasticity, swelling and shrinkage, as well as the plant morphogenesis. In addition, a derivation of von Kármán equations with the assumption of neglected feedback from deflection to 'Growth' was obtained. The growth strain tensor and growth curvature tensor purposed by Lewicka et al. instead of external load were then believed to be the real causes of the deflection and residual stress, and these could be described by incompatible von Kármán equations. Sharon et al. [21] demonstrated this point of view through interesting tests, showing that growing leaves and plastically strained ribbons can be relaxed to different shapes when they were cut in different directions for the partial relief of incompatible strains.

With the identification of the above essence of post buckling problem, the pure numerical method is widely used to investigate the relationship between residual stress and deflection. Yukawa and Ishikawa [25] determined the buckling critical load and post buckling deflection with arbitrary form of residual stress by finite element method (FEM). Abdelkhalek et al. [1–3] investigated a completely coupled approach to simulate the stress profiles and flatness defects with a simple buckling criterion using specialized FEM software Lam/Tec3 and uncoupled asymtotic numerical method (ANM), rendering excellent computational buckling capability and more realistic results. Qin et al. [19] estimated the buckling critical load and wave configuration of oblique and herringbone buckling with the spline FEM, pointing out shear stress being the main reason for oblique and herringbone buckle due to residual strains from the rolling process or applied non-uniform loading. Kpogan et al.

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[13] present a numerical technique to model the buckling of a rolled thin sheet within the Arlequin framework, a three dimensional model based on 8-nodes tri-linear hexahedron. The resulting nonlinear problem is solved by the ANM.

In light of the non-linear nature of post buckling problem, semi-analytical method is also a common way, including the energy vairation method (Timoshenko method, Ritz method) on the basis of the minimum potential energy principle. Timoshenko and Gere [22] first adopted the energy vairation method to determine the buckling and post-bucking of plates under uniform external in-plane compressions. Yang [24], Fischer et al. [10–12] and Rammerstorfer et al. [20] set up a groups of typical distribution of residual stresses (or corresponding membrane forces), and predescribed polynomial form along transverse direction and harmonic form along rolling direction of buckling deflection, and calculated coefficients of deflection form through Ritz method. Nakhoul et al. [16] established a multi-scale buckling model that considers the deflection variation along rolling direction obtained from von Kármán equations solution in energetic formultion.

Noteworthily, the prebuckling problem of thin plates can be treated as a boundary layer problem [9] that involves reduced order of governing equations when the small parameter becomes zero. This problem can be solved by singular perturbation methods, including matched expansion method, composite expansion method [5,6], multiple-scales method and WKB method, which have been summarized in detail by Nayfeh [17].

To overcome the limitation and complexity of matched expansion method, the composite expansion method is applied to analyze the post buckling problem of infinite thin rolled strip with only incompatible strain. This method was first proposed by Chien [5] to approximate the deflection and membrane stress of clamped thin circular plate under uniform normal pressure. Based on the Hencky's solution for the circular membrane, the attached correction term which is dominant in a narrow region close to the edge of the plate, was constructed to satisfy all boundary conditions. This work was highly improved by applying an unknown small parameter by Chien and Chen [6]. O'Malley [18] rediscovered the composite expansion method independently, dealt with a fundamental nonlinear initial value problem and showed how the method could be extended to similar problems for systems with two small parameters and for differential-difference equations with small delays. Bakri et al. [4] recently applied so-called O'Malley-Vasil'eva expansion method [23] to obtain an aymtotic approximation for the modified one-dimenssion Liouville–Bratu–Gelfand nonlinear two-point boundary value problem.

In this paper, the relation of residual stress (in-plane state) and incomaptibilty tensor was first established in incompatible von Kármán equations. Then composite expansion method was first applied to create the rolled strip's deflection in a flexible form with unknown wave length and unknown positions of boundary layers. These unknown geometry parameters were calculated by Ritz method after eliminating the stress potential by deflection(using the method of constant variation). Actual measured strip's residual stress and deflection are practically used to verify the accuracy of composite expansion-Ritz method.

2. Overview of the model

The basic geometric and physical model are first presented as below. Consider an infinite homogeneous isotropic elastic strip of thickness h and width b with Young's modulus E and Poisson's ratio v, whose mid-plane occupies the domain

$$\Omega \equiv \left\{ (x,y) \in \mathbb{Z}^2 \middle| 0 < x < b, -\infty < y < \infty \right\}.$$

Here, x is the transverse direction with unit vector $\boldsymbol{i}$ and y is the rolling direction with unit vector $\boldsymbol{j}$. Considering the continuous processing, the range of variable y is set to be infinity. While the actual detected residual stress component

$$\bar{\sigma}_y \equiv \bar{\sigma}_y(x,y)$$

is self-equilibrating in pre-buckling state (relative to σ_y in post buckling state).

2.1. Boundary value prolem for post buckling of rolled strip

The governning equations to post buckling of incompatible thin plates have been clearly derived by Zubov [26] and Lewicka et al. [15], which can be written as:

$$\begin{cases} \dfrac{D}{h}\Delta^2 w - L(\Phi,w) = \dfrac{p}{h} \\ \dfrac{1}{E}\Delta^2\Phi + \dfrac{1}{2}L(w,w) = 0 \end{cases} \text{by von K∧rm∧n,} \tag{1}$$

$$\begin{cases} \dfrac{D}{h}\Delta^2 w - L(\Phi,w) = \dfrac{p}{h} \\ \dfrac{1}{E}\Delta^2\Phi + \dfrac{1}{2}L(w,w) = \mu \end{cases} \text{by Zubov,} \tag{2}$$

$$\begin{cases} \dfrac{D}{h}\Delta^2 w - L(\Phi,w) = \dfrac{D}{h}\left[\nu\left(\nabla\times\boldsymbol{\kappa}_g\times\nabla\right) : (\boldsymbol{kk}) - \nabla\cdot\boldsymbol{\kappa}_g\cdot\nabla\right] \\ \dfrac{1}{E}\Delta^2\Phi + \dfrac{1}{2}L(w,w) = \left(\nabla\times\boldsymbol{\varepsilon}_g\times\nabla\right) : (\boldsymbol{kk}) \end{cases} \text{by Lewicka.} \tag{3}$$

Here, $D = Eh^3/12(1-\upsilon^2)$ is the bending stiffness of plate, $\boldsymbol{k}=\boldsymbol{i}\times\boldsymbol{j}, w \equiv w(x,y)$, $\Phi \equiv \Phi(x,y)$ and $p \equiv p(x,y)$ represent deflection, stress potential and normal external load of plate respectively.$\mu \equiv \mu(x,y)$ defined by Zubov [26] is a metric of defects. Growth strain tensor $\boldsymbol{\varepsilon}_g \equiv \varepsilon_x^g(x,y)\boldsymbol{ii} + \varepsilon_{xy}^g(x,y)\boldsymbol{ij} + \varepsilon_{yx}^g(x,y)\boldsymbol{ji} + \varepsilon_y^g(x,y)\boldsymbol{jj}$ and growth curvature tensor $\boldsymbol{\kappa}_g \equiv \kappa_x^g(x,y)\boldsymbol{ii} + \kappa_{xy}^g(x,y)\boldsymbol{ij} + \kappa_{yx}^g(x,y)\boldsymbol{ji} + \kappa_y^g(x,y)\boldsymbol{jj}$ have been mentioned. Operators involved in (1)–(3) are expressed as: the two-dimensional Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the two-dimensional Hamilton operator $\nabla = \partial/\partial x\boldsymbol{i} + \partial/\partial y\boldsymbol{j}$, and the Airy 's bracket $L(\Phi, w) = \Delta\Phi\Delta w - \nabla\nabla\Phi : \nabla\nabla w$, where ':' is the double-scalar product operator, similarly, $L(w, w) = (\Delta w)^2 - \nabla\nabla w : \nabla\nabla w$.

The second equation in Eq. (1) is so-called compatibility equation, and is only suitable for thin elastic plate bending under normal external load. Aiming at post buckling of thin cold rolled strip without any external load, we have

$$\begin{cases} \dfrac{D}{h}\Delta^2 w - L(\Phi,w) = 0 \\ \dfrac{1}{E}\Delta^2\Phi + \dfrac{1}{2}L(w,w) = (\nabla\times\boldsymbol{\varepsilon}^p\times\nabla) : (\boldsymbol{kk}) = I \end{cases} \quad in \quad \Omega, \tag{4}$$

where $\boldsymbol{\varepsilon}^p \equiv \varepsilon_x^p(x,y)\boldsymbol{ii} + \gamma_{xy}^p(x,y)\boldsymbol{ij} + \gamma_{yx}^p(x,y)\boldsymbol{ji} + \varepsilon_y^p(x,y)\boldsymbol{jj}$ is two-dimensional plastic strain tensor, which is generated by rolling and can be treated as an initial strain in post buckling analysis, $I \equiv I(x, y)$ is a known scalar function relate to plastic strain. The first equation in Eq. (4) is obviously tenable when there is no initial curvature in strip. Meanwhile, the second equation in Eq. (4) can be obtained based on the theory proposed by Kröner [14] and Lewicka et al. [15] where plastic strain is used to replace growth strain tensor specifically. As is known to all, the relevant boundary conditions in edges are

$$\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2} = 0,\ \frac{\partial^3 w}{\partial x^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x\partial y^2} = 0,\ \frac{\partial^2\Phi}{\partial y^2} = 0,$$
$$\frac{\partial^2\Phi}{\partial x\partial y} = 0 \text{ in } x = 0, b. \tag{5}$$

2.2. Simplification of boundary value prolem under some assumptions

In order to figure out the property of incompatibility in rolling, we consider the situation of trivial solution $w \equiv 0$. The first equation in Eq. (4) is automatically satisfied, the second one, under the assumption that the feedback from deflection to incompatibility is ignored [15], can be written as

$$\frac{1}{E}\Delta^2\bar{\Phi} = I \qquad \text{in} \quad \Omega, \tag{6}$$