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Safe regions with partial control of a chaotic system in the presence of white Gaussian noise

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ABSTRACT

Noise is present in a wide variety of engineering systems, and it can play a significant role in influencing system dynamics. Under the influence of noise, it has been shown that the response of many discrete-time dynamical systems can be moved away from a particular region. In the present study, the partial control scheme constructed for a chaotic system is applied for confining the trajectories inside a particular region despite the presence of white noise. The proposed algorithm is independent of the dimension of the system. As an illustration, the partial control method has been applied to restrict the response of a Duffing oscillator to a certain state-space region. Different noise forms are considered and numerical results are presented to illustrate the effectiveness of this control method.

1. Introduction

In recent years, the effects of noise on the response of a Duffing oscillator have received considerable attention (e.g., [1–4]). Generally speaking, one considers noise as being undesirable for the performance of an engineering system. Chaotic motions of a system may also not be preferrable and various means have been used to control these motions. The proceedings of a IUTAM meeting organized by Professors Rega and Vestroni serves as a rich example [5]. Aperiodic behavior can play a significant role in influencing system dynamics, as observed in the context of several systems; for example, avoidance of undesired resonances in mechanical systems and enhancement of the efficiency of a thermal pulse combustor are some applications where chaos can be beneficial [6,7]. There are also other no-mechanical systems such as biological systems, in which chaotic behavior can play an important role. In living organisms, chaotic dynamics has been stated to be important for some vital functions [8]. It has been noted that preserving chaos is of potential relevance to biological disorders [9].

There are certain situations in which a system's trajectory has characteristics of chaotic behavior for a finite duration of time, before the trajectory escapes to another state that could correspond to a periodic attractor or another state. Sometimes, this end state could be undesirable. This type of behavior is often described as transient chaos. An important example of this kind of behavior is undesired tumor growth [10].

Maintaining chaotic behavior in systems in the presence of an external disturbance can be desirable and important for the dynamics of the considered system. This has motivated studies and efforts on different control techniques, such as the partial control method [11–13]. These methods have been designed for application to deterministic systems with bounded noise [14]. The presence of even a low level of noise can radically change the dynamics of a chaotic system, as these perturbations can experience exponential growth.

In the method proposed by Sabuco et al. [12], the property of transient chaos is used. The goal is to keep the trajectory inside a particular region without moving towards any attractor. The method has been shown to be effective, and one can use this method to control the trajectory by using an upper bound on the control u_0 that is less than the upper bound on the disturbance ξ_0. This control method has been applied to several dynamical systems, including the Hénon map, Duffing oscillator, and other systems in the context of ecology and cancer [10,15]. In all of the previous use of partial control methods, a bounded representation of noise has been used. To date, the control of trajectories of a system in the presence of white noise has not been studied. This is addressed in this work and the development of the partial control method in the presence of a white noise disturbance represents a fundamental difference between this study and the previous studies on partial control reported in the literature.

The rest of this paper is organized as follows. In Section 2, the authors describe the concept of a safe set, the Sculpting Algorithm for computing a safe set, and the partial control method for a system with white noise by using the Euler-Maruyama integration method. In

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Section 3, an application of this method to the response of a Duffing oscillator is considered. In particular, parameters for which this system experiences a transient chaotic behavior are considered. In Section 4, comparisons are made between the cases with white noise and bounded noise. Concluding remarks are collected together and presented in Section 5.

2. Partial control in the presence of white noise

2.1. Escaping trajectories

Let f be a continuous map of phase space; then, one can write

$$q_{n+1} = f(q_n), \tag{1}$$

where the trajectory at $(n+1)$ th step is mapped to the nth step. In nonlinear systems, for a given choice of parameters, the trajectories may exhibit chaotic behavior for a while before eventually leaving that particular region or reaching a stable periodic state. As previously mentioned, this behavior is referred to as "transient chaotic" behavior, and the topological structure inside region Q associated with transient chaotic behavior is a zero-measure set known as a chaotic saddle [16,17].

In various practical applications, due to external disturbances, trajectories typically rapidly leave the region of the phase space where transient chaos occurs. To model this, the authors consider that there is a white noise component $\sigma\dot{W}(t)$ that causes the trajectory q_n to leave the region Q, where by leaving a region, the authors mean that the trajectory is leaving that particular region Q, or it converges towards a fixed point or an attractor that one does not consider to be part of that particular region Q. In equation form, the relevant map can be written as

$$q_{n+1} = f(q_n) + \xi_n. \tag{2}$$

Here, q_n is the state at step n, f is a function with chaotic transient in Q, and ξ_n is the noise input. The noise input to the system is represented by $\sigma\dot{W}(t)$, where σ represents noise amplitude, $W(t)$ represents the Wiener process and $\dot{W}(t)$ is the associated derivative of Brownian motion. The authors' goal is to choose a control u_n such that for the partially controlled trajectories governed by

$$q_{n+1} = f(q_n) + \xi_n + u_n, \tag{3}$$

one can guarantee that the q_n remain in region Q for an appropriate choice of control u_n with an upper bound of u_0. The authors refer to u_n as feedback control that can be chosen with the knowledge of white noise component $\sigma\dot{W}(t)$ and $f(q_n)$, or in particular $f(q_n) + \xi_n$. Therefore, the goal is to find an appropriate feedback control u_n which is a function of $f(q_n) + \xi_n$, and is bounded by u_0. It is also worth noting that the applied control u_n is a discrete control input. To apply this control method to a continuous dynamical system, one has to consider a discretized state of the system. This is usually done by constructing a Poincaré section for autonomous systems, or a stroboscopic map for nonautonomous systems such as forced oscillators. With this discretization, the control input required by the partial control method u_n is only applied discretely at regular intervals of time (when stroboscopic maps are used) or when the trajectory crosses the defined Poincaré section. At any other instant, the system is allowed to evolve freely, without any kind of control.

In the present work, the following assumptions are made:

1. The region Q is a closed and bounded region in the phase space.
2. The applied feedback control u_n in phase space has an upper bound u_0, which means that it satisfies $|u_n| \le u_0$. Such control u_n is called "admissible control". Here, only admissible control is considered.
3. The bound on the control u_0 depends on the noise amplitude σ.

Here, a safe set S is defined as the set of points in a bounded region, satisfying the following:

1. A safe set S is a subset of Q; that is, $S \subset Q$.
2. For each point q_n in phase space S $(q_n \in S)$, the distance of $f(q_n) + \xi_n$ from S is at most u_0. This implies that there exists an admissible control value u_n which has an upper bound u_0, such that $f(q_n) + \xi_n + u_n$ is in S, or $f(q_n) + \xi_n + u_n \in S$. A safe set is decided by the white noise amplitude σ and control bound u_0.

By applying admissible control, it is possible to keep the entire trajectory q_n of Eq. (3) in S and hence in Q. Then, if q is in a safe set $S \subset Q$, the trajectories can be controlled to stay in S and consequently in Q by choosing the control u_n so that q_{n+1} is in S. For a bounded disturbance, wherein the disturbance ξ_n is bounded by ξ_0, the trajectories are allowed to be remain inside a region Q even when the upper bound of the control u_0 is smaller than the upper bound of the disturbance ξ_0 [12]. In prior work, safe sets have only been found for one-dimensional and two-dimensional maps for bounded noise values of $\xi_n \le \xi_0$ [12,18].

In the current study, while applying the partial control method, a grid of points is used for the close bounded region Q that needs to be controlled, and the largest safe set S is found.

2.2. Form of safe sets

Over the last few years, researchers have considered cases with bounded noise and found safe sets for these cases [12,18]. The algorithm for finding a safe set for a bounded noise is available, and it is known that the shape of a safe set can be geometrically more complicated than expected [12].

In the present work, the authors have followed the steps used by Sabuco et al. [12]. However, the authors have had to change the numerical integration scheme, since the authors have considered white Gaussian noise instead of bounded noise.

In the prior section, the authors mentioned important properties of a point belonging to a safe set. These properties will be used to develop the algorithm to compute safe sets by using a recursive algorithm. The algorithm is based on the Euler-Maruyama integration scheme and it can be used to find a safe set whenever there is a chaotic saddle in the region Q. The algorithm has been demonstrated on a Duffing oscillator that shows the Wada property which arises in the phase space for all the basins of attraction [19]. There also exist fixed points and periodic attractors inside the region Q, and almost all trajectories eventually are attracted to one of the fixed points or periodic attractors, as it is expected for cases with transient chaos.

2.3. Sculpting algorithm for computing largest set

Consider a closed bounded region Q represented by a set of grid points that has to be controlled. Again, the trajectory of any point q in Q without any disturbance is given by

$$q_{n+1} = f(q_n). \tag{4}$$

The trajectories are associated with transient chaotic behavior. The application of a white Gaussian noise $\sigma\dot{W}(t)$ is represented as

$$q_{n+1} = f(q_n) + \sigma dW. \tag{5}$$

Given the closed bounded set Q and upper bound of control u_0, it is declared that a point q in Q "unsafe" (for Q) under the influence of white noise $\sigma\dot{W}(t)$ if the distance of $f(q_n) + \sigma dW$ from region Q is more than the upper bound of the control u_0, and such $f(q_n) + \sigma dW$ has no admissible control u for which $f(q_n) + \sigma dW + u$ is in Q. Then, the authors define the sculpting operation Υ that results in the removal of the unsafe points from Q. This implies that $\Upsilon(Q)$ is the set of safe points in Q after removal of the unsafe points.

The authors start by considering a grid of points in a close bounded