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Non-linear argumental oscillators: Stability criterion and approximate implicit analytic solution

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ABSTRACT

The behaviour of a space-modulated, so-called "argumental" oscillator, is studied. The oscillator is submitted to an external harmonic force, which is amplitude-modulated by the oscillator's position in space. An analytic expression of a stability criterion is given. Using the averaging method, an integrating factor and a Van der Pol representation in the (amplitude, phase)-space, an exact implicit analytic solution is given when there is no damping, and an approximate implicit analytic solution is given when there is damping, allowing the plotting of the separatrix curve. An attractor is identified.

1. Introduction

In the 1920s, physicists were searching for a device to divide the mains current frequency in order to manufacture mains-driven clocks. As no electronics were available, they studied various inherently frequency-dividing oscillators. Among them was a pendulum designed by Béthenod in 1929 [1], that oscillated at a low frequency, typically 1 Hz, when driven by the mains at 50 Hz. Béthenod's pendulum was fitted with a steel sphere at the tip of the rod. The force, which could only be attractive, was due to a magnetic field created by a solenoid with vertical axis, carrying an alternating current. The sphere could sense this force only when it was near the lower equilibrium position of the pendulum. Thus, there was a spatial modulation of the force.

An oscillator subjected to a spatially-localized external harmonic force is presented in [2], where the term "argumental oscillations" is coined from the fact that the interaction between the oscillator and the excitation depends on the "argument" of a space-localization function, which is called the H-function hereinafter.

A pendulum fitted with a permanent magnet at the tip of the rod, which can sense the external electromagnetic force only when it is near a coil located at the lower equilibrium position, is presented in [3]. Here the force can be both attractive or repulsive. A model of the spatial localization of the interaction is built, and mathematical elements regarding the system are given.

Argumental oscillations are presented in [4]. An electronic argumental oscillator with a Π function used as dependent-variable localization function is presented in [5].

Modeling and experimental results about six argumental oscillators are given in [6]. Capture probability by an attractor in argumental oscillators is studied in [7].

A system with a second-order equation exhibiting a cubic and a quadratic nonlinearities with an excitation frequency two or three times the system's natural frequency is studied in [8].

The purpose of this article is to study symbolically some aspects of the argumental oscillators, namely a stability criterion and an approximate implicit equation of the integral curves, to be able to draw separatrix lines and assess some areas around the attractors in the (amplitude, phase)-space.

2. Canonical second-order equation of motion

To simplify the expression of the system behaviour, the reduced time $\tau = \omega_0 t$ is classically introduced, where ω_0 is the natural angular frequency of the oscillator. Using from now on the dot notation to refer to the derivatives with respect to τ, we shall distinguish three types of oscillator, which we call "Type A", "Type B1″ and "Type B2″. The general second-order equation of motion for these oscillators is:

$$\ddot{\alpha} + 2\beta\dot{\alpha} + \alpha + \mu\alpha^3 = AH(\alpha)E(\tau) \tag{1}$$

where β is the dissipation coefficient, μ is the Duffing coefficient, A is a constant, H is a function of α, and $E(\tau)$ is a periodic function of time τ, with frequency components located above the oscillator's fundamental frequency.

With ν being the angular frequency of the external excitation:

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- For Type A oscillators, H is an odd function of α, and E is the function $\sin^2\left(\frac{\nu}{\omega_0}\tau\right)$.
- For Type B oscillators, H is an even function of α, and E is the function $\sin\left(\frac{\nu}{\omega_0}\tau\right)$.

An example of a Type-A H-function is

$$H(\alpha) = \frac{\alpha}{(1+\gamma\alpha^2)^3}, \tag{2}$$

where γ is a constant.

An example of a Type-B H-function is

$$H(\alpha) = \Pi\left(\frac{\alpha}{2h}\right), \tag{3}$$

where h is a constant, and Π denotes the Pi function, i.e. $\Pi(x) = 1$ if $|x| < \frac{1}{2}$ and $\Pi(x) = 0$ otherwise. In this paper, this example is called a Type B1 oscillator.

Another example of a Type-B H-function is

$$H(\alpha) = \frac{1-\gamma\alpha^2}{(1+\gamma\alpha^2)^2}, \tag{4}$$

with γ being a constant. In this paper, this example is called a Type B2 oscillator.

The "Type A" oscillator in this paper is a Béthenod's pendulum, or Type II-1 oscillator as discussed in [6]. The Type B1 is a Doubochinski's pendulum, as discussed in [3]. The "Type B2″ oscillator is analogous to a Doubochinski's pendulum, but with a smoother H-function. [3] used a coarser approximation to the H-function, as shown by Eq. (3). This approximation was sufficient to elaborate averaged equations and to derive an expression of the amplitude of the external force as a function of the oscillator's amplitude, leading to an explanation of a discrete set of stable amplitudes. We use herein our smoother and more precise Type-B2 H-function, as shown by Eq. (4), with the advantage of handling a H-function which is C^∞: this will allow to eliminate artefacts in a function relative to the Type-B1 model, and to derive an approximate symbolic solution.

It will be shown later on that after the averaging process, the systems of equations of Types A, B1 and B2 are formally similar.

3. Calculus workflow

Having available the reduced-time second-order differential equation of motion for both oscillators, it is classically considered that a perturbation method could be an appropriate approach, because the oscillator is almost always in a free-run mode. Only at certain narrow locations in space will it "feel" the external force. Moreover, this force is of small amplitude. Keeping the expressions under symbolic form, we shall go through three steps to get to an analytic approximation of the solution to Eq. (1).

The averaging method used in the first and second steps is classical, and has been described in [9], and used in [3]. So only the implementation of the method will be outlined for these two steps. Our contribution to the second step is the symbolic expression of the Fourier series for the H-functions of the Type-A and Type-B oscillators, the symbolic expression of the stability criterion in the general case, and the Van der Pol polar representation of the averaged amplitude and phase. Our contribution further consists of the third step, which will be detailed hereinafter.

- The first step of the calculus is to replace the second-order differential equation of motion by two first-order equations to get the classical standard system of equations.
- The second step is to form a Fourier series of the H-function and to apply the averaging method to obtain an averaged system of equations. As the external force is periodic, simplifications can be expected.
- The third step is to find an integrating factor to approximately solve the averaged system, while keeping the symbolic form of the equations.

4. First step: building the standard system of equations

Starting from the equation of motion under its general form as shown by Eq. (1), define a function X by:

$$X(\tau, \alpha, \dot{\alpha}) = -2\beta\frac{d\alpha}{d\tau} - \mu\alpha^3 + AH(\alpha)E(\tau). \tag{5}$$

Thus Eq. (1) can be rewritten:

$$\frac{d^2\alpha}{d\tau^2} + \alpha = X(\tau, \alpha, \dot{\alpha}). \tag{6}$$

By observing the experimental oscillators [6] and corresponding numerical simulations, one concludes that the motion is close to that of a free-running oscillator, with slowly varying amplitude and phase. Hence the slow-varying amplitude $a(\tau)$ and phase $\varphi(\tau)$ are introduced as two new independent variables, which will replace the variables α and $\dot{\alpha}$. The motion, expressed as a function of t or τ, will be $\alpha(t) = a(t)\sin(\omega t + \varphi(t)) = a(\tau)\sin(\rho\tau + \varphi(\tau))$, where ω is a parameter close to ω_0, and $\rho = \frac{\omega}{\omega_0}$. As these two new independent variables are chosen, we found it natural to introduce a Van der Pol representation, with a as abscissae and φ as ordinates. Alternatively, a polar Van der Pol representation will also be used, i.e. a as radius and φ as angle.

Define the change of variables by putting:

$$\alpha(\tau) = a(\tau)\sin(\rho\tau + \varphi(\tau)), \tag{7}$$

$$\dot{\alpha}(\tau) = a(\tau)\rho\cos(\rho\tau + \varphi(\tau)). \tag{8}$$

This is natural, because Eq. (8) is obtained by differentiating Eq. (7) with a and φ taken as constant. This is simply the implementation of the physical observation [6] that a and φ vary slowly with respect to the period of the free-running oscillator.

Differentiating Eq. (7) and comparing the result with Eq. (8) yields:

$$\dot{a}\sin(\rho\tau + \varphi) + a\dot{\varphi}\cos(\rho\tau + \varphi) = 0 \tag{9}$$

Differentiating Eq. (8) and putting the result into Eq. (6) yields:

$$\dot{a} = \frac{\cos(\theta)}{\rho}(X(\tau, a\sin(\theta), a\rho\cos(\theta)) + a\sin(\theta)(\rho^2 - 1)), \tag{10}$$

where $\theta = \rho\tau + \varphi$. This is the first differential equation involving only the two new variables a and φ.

In Eq. (10), replacing $X(\tau, a\sin(\theta), a\rho\cos(\theta))$ by its expression given in Eq. (5) yields, taking into account Eqs. (7) and (8):

$$\begin{aligned}\dot{a} &= \frac{\cos(\theta)}{\rho}\left(-2\beta\dot{\alpha} - \mu\alpha^3 + AH(\alpha)\sin\left(\frac{\nu}{\omega_0}\tau\right) + a\sin(\theta)(\rho^2 - 1)\right)\\ &= \frac{\cos(\theta)}{\rho}\left(-2\beta a\rho\cos(\theta) - \mu a^3\sin^3(\theta) + AH(a\sin(\theta))\sin\left(\frac{\nu}{\omega_0}\tau\right)\right.\\ &\quad \left. + \; a\sin(\theta)(\rho^2 - 1)\right).\end{aligned} \tag{11}$$

Substituting in Eq. (9) the expression (10) for $\dot{a}$ yields:

$$\dot{\varphi} = -\frac{1}{a\rho}(a\sin(\theta)(\rho^2 - 1) + X(\tau, a\sin(\theta), a\rho\cos(\theta)))\sin(\theta) \tag{12}$$

which is the second differential equation involving only the two new variables a and φ. From Eq. (12), it is obtained, in the same way as for Eq. (11):