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On the design of external excitations in order to make nonlinear oscillators respond as free oscillators of the same or different type

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ABSTRACT

This study deals with nonlinear oscillators whose restoring force has a polynomial nonlinearity of the cubic or quadratic type. Conservative unforced oscillators with such a restoring force have closed-form exact solutions in terms of Jacobi elliptic functions. This fact can be used to design the form of the external elliptic-type excitation so that the resulting forced oscillators also have closed-form exact steady-state solutions in terms of these functions. It is shown how one can use the amplitude of such excitations to change the way in which oscillators behave, making them respond as free oscillators of the same or different type. Thus, in cubic oscillators, a supercritical or subcritical pitchfork bifurcation can appear, whilst in quadratic oscillators, a transcritical bifurcation can take place.

1. Introduction

This study is concerned with externally excited nonlinear oscillators governed by

$$\ddot{x} + c_1 x + c_\alpha x^\alpha = F(t), \tag{1}$$

where x is the displacement, c_1 and c_α are the coefficients of the linear and nonlinear stiffness terms, and where c_α is not necessarily small, while α is the power of nonlinearity that can be equal either to 3 or 2, and, thus, results in a cubic or quadratic nonlinearity; the overdots denote differentiation with respect to time t and $F(t)$ is an external periodic excitation.

Systems that are approximately or exactly governed by Eq. (1) appear widely in physics and engineering, and some of them are: pendula, snap-through mechanisms, beams, cables, human eardrum oscillations, vibration isolators, etc. (see, for example, [1] and the references cited therein). Given this wide range of applications, obtaining their steady-state response to external periodic forcing has been of particular interest and has resulted in the development of many analytical techniques to find approximate steady-state responses [2–4]. However, the aim here is to show how to design the periodic excitation $F(t)$ to get an *exact* analytical steady-state solution, noting that these are normally very scarce in Nonlinear Dynamics. The concept of the “exact steady state” of a strongly non-linear, undamped, discrete system was defined by Rosenberg [5,6]: for the steady state forced response of a single degree of freedom the ratio of the response and the amplitude is “cosine-like” [6] and of the same period of that of the periodic forcing function. Harvey considered “natural forcing functions” proportional to the nonlinear restoring forces and applied them to the study of the forced Duffing problem [7]. Caughey and Vakakis [8] examined the exact steady states of a certain class of strongly nonlinear systems of two degrees of freedom. By expressing the forcing as a function of the steady state displacements, the forced problem was transformed to an equivalent free oscillation and subsequently a matching procedure was followed which resulted in the uncoupling of the differential equations of motion at the steady state.

The basic idea used in this work dates back to Hsu's paper [9], in which he considered Duffing-type oscillators ($\alpha = 3$) governed by Eq. (1) with a positive c_1 and a positive or negative c_3. As these oscillators have exact closed-form solutions for the conservative unforced case expressible in terms of Jacobi elliptic functions, Hsu's approach led to the external excitation having the same form, i.e. being proportional to the displacement and being expressed in terms of Jacobi elliptic functions. This idea is extended in this work to all other oscillators with the cubic or quadratic nonlinearities that have exact closed-form solutions for the conservative unforced case.

This paper is organised as follows. For the sake of the reader the first part of Section 2 contains an overview of exact closed-form solutions for certain nonlinear oscillators with cubic and quadratic

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nonlinearity that are expressed in terms of Jacobi elliptic functions and depend on the system parameters and the amplitude. The second part of Section 2 includes a brief outline of Hsu's approach for hardening and softening Duffing oscillators with some physical interpretations in terms of the forcing amplitude factor. In Section 3, exact solutions for other nonlinear oscillators with elliptic-type external excitation are derived for the first time. They include: bistable oscillators in full swing mode and half-swing mode as well as pure cubic oscillators. In Section 4, bifurcations in cubic quadratic oscillators are investigated. It is shown analytically and confirmed numerically how one can design the external excitation with respect to the parameters of these oscillators to change the way in which the excited oscillators behave, making them respond as free oscillators of the same or different type. The corresponding types of bifurcation are also discussed.

2. Known exact solutions for free and forced oscillations

2.1. Exact solutions for free oscillations

Several nonlinear oscillators governed by

$$\ddot{x} + c_1 x + c_\alpha x^\alpha = 0, \tag{2}$$

have an exact closed-form solution for their free response in terms of Jacobi elliptic functions. These oscillators are [1,10]:

- Hardening Duffing Oscillator (HDO), for which $c_1 > 0,\ c_3 > 0,\ \alpha = 3$;
- Softening Duffing Oscillator (SDO), for which $c_1 > 0,\ c_3 < 0,\ \alpha = 3$;
- Bistable Duffing Oscillator (BDO), for which $c_1 < 0,\ c_3 < 0,\ \alpha = 3$, where two cases can be recognised. The first one is labelled here by BDO1 and represents the so-called full swing mode (motion surrounding all the equilibria), and the second one is labelled here by BDO2 and represents the so-called half swing mode (motion surrounding one of the non-zero equilibria);
- Pure Cubic Oscillator (PCO), for which $c_1 = 0,\ c_3 < 0,\ \alpha = 3$;
- Quadratic Oscillator (QO), for which $\alpha = 2$.

All of these solutions are expressed in terms of Jacobi elliptic functions (cn, sn or dn) and are listed in Table 1 for each oscillator. Note that Jacobi elliptic functions have two arguments. In the first one, the frequency ω appears [1,11]. The second argument is the elliptic parameter m [1,11], and ranges from 0 to 1 (other values can also exist, but require certain transformations of the original Jacobi elliptic functions, and are, thus, avoided here). Note also that, instead of the elliptic parameter, one can use the elliptic modulus $k^2=m$. The value $m = 0$ transforms the cn function into the Cosine function, the sn function into the Sine function, while the dn function becomes equal to unity. As seen from Table 1, both the frequency ω and the elliptic parameter m depend, in general, on the stiffness coefficients and the amplitude, while in the case of the PCO, the elliptic parameter is constant.

The only specific case in Table 1 is the QO and this includes several features. First, it is the only oscillator from the list whose response is the quadratic function of the elliptic functions. Second, unlike other oscillators whose elliptic parameter is the explicit single-valued function of the system parameter and the amplitude, this parameter m is implicitly defined here by $A = c_1\left(m + 1 - \sqrt{m^2 - m + 1}\right)/\left(2c_2\sqrt{m^2 - m + 1}\right)$ [12], although this expression can be transformed further to get a real value of m. In addition, when $c_1 < 0$, one has $\omega = 0.5\sqrt{c_1}/(m^2 - m + 1)^{1/4}$, and it is obvious that the frequency becomes complex. However, in the case of complex arguments of Jacobi elliptic functions, certain transformations can be used to get real arguments [11]. Contemporary computer algebra and symbolic software packages usually have these transformations built-in, offering improvements in ease of computation and transformations.

Table 1 also includes typical phase planes for all the oscillators listed with the trajectories surrounding their equilibrium/equilibria (stable equlibria are depicted by the black dots and the unstable ones by the white dots).

All these solutions are closed-form, but to explain and understand what kind of functions they actually represent, one can use the corresponding Fourier series expansions (see the Appendix A):

$$\mathrm{cn}(t|m) = \sum_{N=1}^{\bar{N}} C_N \cos\left[(2N-1)\frac{\pi}{2K}t\right], \tag{3a}$$

$$\mathrm{sn}(t|m) = \sum_{N=1}^{\infty} S_N \sin\left((2N-1)\frac{\pi}{2K_r}t\right), \tag{3b}$$

$$\mathrm{dn}(t|m) = D_0 + \sum_{N=1}^{\infty} D_N \cos\left(N\frac{\pi}{K}t\right), \tag{3c}$$

whose amplitudes depend on m, i.e. $C_N = C_N(m)$, $S_N = S_N(m)$, $D_0 = D_0(m)$, $D_N = D_N(m)$. As can be seen, all of them can be interpreted as multi-term periodic excitations; the cn and sn functions contains odd harmonics, while the dn function contains both the offset and odd and even harmonics; in all cases, the amplitude and frequencies of the harmonics depend on the elliptic parameter and are, thus, mutually related, as defined in the Appendix A.

2.2. Brief outline of Hsu's approach with new interpretations

Hsu considered the following periodically driven Duffing oscillator [9]

$$\ddot{x} + c_1 x + c_3 x^3 = F(t), \tag{4}$$

including cases when c_1 is positive, while the coefficient c_3 can be either positive (HDO) or negative (SDO). The key point of Hsu's approach is to transform this nonautonomous system into an autonomous one, and then, for such a system, to utilise known expressions for the exact solutions. To that end it is assumed that the response x and the excitation force F are proportional, i.e. $F=B\,x$. With this assumption, Eq. (1) becomes

$$\ddot{x} + (c_1 - B)x + c_3 x^3 = 0. \tag{5}$$

It can be seen that Eq. (5) corresponds to the autonomous system (2) whose exact solutions are given in Table 1. It is important to point out that the sign of the coefficient in front of the linear term $(c_1 - B)$ now depends on the parameter B. For the time being it is assumed that $c_1 > B$, if not noted differently, while other cases are analysed in Section 4.

2.2.1. Forced HDO

The first case considered is when the transformation $F=Bx$ is applied to the HDO, while the assumption $c_1 > B$ is retained. The resulting equation also corresponds to the HDO, and the exact closed-form solution can readily be found based on those given in Table 1. Two relationships for the parameters given in Table 1 for the HDO now become

$$\omega_r^2 = c_1 - B + c_3 A^2, \tag{6a}$$