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A bipotential approach for plastic limit loads of strip footings with non-associated materials

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ABSTRACT

This paper is concerned with a bipotential approach for estimating the plastic collapse loads of a half-space made with a non-associated Mohr–Coulomb material and indented by a rigid punch. In geotechnics, this problem is called the bearing capacity of shallow strip footing for which the analytical solution is derived by Prandtl (1920) [46] and Hill (1950) [35] in the context of associated plasticity. However, when the plastic model is not associated, no analytical methods have yet been developed. Here we explore this issue in a rigorous mathematical framework coupling the bipotential concept and limit analysis. First, the method proposed makes use of the method of characteristics to build a statically and plastically admissible stress field that enables a lower estimate of the plastic limit loads. Next, the extended kinematic theorem of limit analysis to non-standard plasticity is applied to derive an upper quasi-bound of the collapse loads. For this aim, the internal rate of plastic dissipation is obtained thanks to the bipotential functional depending on both a trial stress field and a Prandtl-like collapse mechanism. The analytic estimates are compared to the formulae and numerical results provided in literature.

1. Introduction

Limit analysis [15,16,50,20] is a powerful method for the direct determination of the collapse load of structures subjected to proportional loadings and operating beyond the elastic limit. The constitutive laws are supposed rigid-perfectly plastic, modelled by a plastic domain and an associated plastic flow rule. Typically, the task is to predict the ultimate load factors using the lower and upper theorems related to static and kinematic approaches respectively.

We recall that the static method is based on a trial stress distribution in internal equilibrium with the applied loading and satisfies the traction boundary conditions and the plastic criterion. The minimum multiple of this distribution that will cause the solid to collapse provides a lower bound of the ultimate load. The dual (kinematic) approach is based on a failure mechanism that involves a kinematically and plastically admissible velocity vector. The rate of change of internal energy balances the power of external loads which assess the upper bound of the plastic limit load. The exact collapse load is obtained when the static and kinematic approaches provide the same value of the limit load.

For a long time, limit analysis has been used to evaluate the bearing capacity of foundations and the slop stability for geomaterials modelled by associated plastic laws. In particular, a problem of practical interest for engineers is the bearing capacity of a semi-infinite soil foundation with the standard law of Mohr–Coulomb, a cohesion c and a friction angle φ. Prandtl–Hill analytical solutions provide the exact limit load [46,35,15].

Another development in computing the bearing capacity of foundations is achieved by using the finite element method and the finite difference method [29,30,40,52,27,47]. Note also that numerical limit analysis bounds involving linear/non-linear programming are proposed in the literature [34,49,36]. A notable advantage of these numerical methods is the study of three-dimensional problems with complex geometries and loadings.

It is noteworthy that the classical limit analysis theorems are restricted to standard materials with associated flow rule (the plastic strain rate is normal to the yielding surface). However, many experimental observations showed that for geomaterials and polymers, the dilatancy angle is lower than the friction one and thus the plasticity is not associated. Decidedly this affects the failure mechanism and the plastic limit loads. The assessment of the closed-form expression of the bearing capacity of a strip footing remains open. Many numerical results are proposed in the literature [16,52,39,33]. Moreover, a simple but widely used formula for computing the limit load has been proposed by Drescher and Detournay [25].

The classical approach for modelling the non-associated constitutive

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laws relies on a dissipation plastic potential different from the plastic yield. An alternative theory is provided by the concept of bipotential introduced by de Saxcé [21] and defining the framework of the Implicit Standard Materials. The application of this concept permits the generalization of the classical variational principles to the boundary value problems with non-standards materials. This formulation combined with the incremental finite element method was applied in [1,2] to study the vertical bearing capacity of the shallow footing with the non-associated Drucker–Prager criterion. Concerning the theory of limit analysis, the rigorous extension of the upper and lower theorems the Implicit Standard Materials has been proposed in [21,23].

The present work provides analytical estimates of the collapse loads of strip footing with the non-associated Mohr–Coulomb model. The paper is organized as follows. In Section 2, a brief review of the basic notions of the bipotential concept and the extension of the limit analysis tools to the non-associated plasticity are presented. In Section 3, we set out the non-standard Mohr–Coulomb and the plastic flow rule. We prove that this criterion belongs to the Implicit Standard Materials class and we build its bipotential functional. Next, in Section 4, we develop the analytic computations for estimating the collapse load by using the slip-line method and the bipotential tools. Section 5 presents a comparison between the analytic estimates of the bearing capacity of the strip against results and numerical computations provided in the literature. The paper concludes by listing the major contributions of this study and some perspectives for future work.

2. The bipotential concept

2.1. Definition and basic relations

Let X be a topological vector space of velocities $\dot{\kappa}$, and Y be its dual space of like-stress variables π, put in duality through a dual pairing $X \times Y \to \mathbb{R}$: $(\dot{\kappa}, \pi) \mapsto \dot{\kappa}\cdot\pi$. In convex analysis, the sub-differential of a function φ in a point $\dot{\kappa}$ is the (possibly empty) set:

$$\partial\varphi(\dot{\kappa}) = \{\pi \in Y | \forall\, \dot{\kappa}' \in X,\, \varphi(\dot{\kappa}') - \varphi(\dot{\kappa}) \geq (\dot{\kappa}' - \dot{\kappa})\cdot\pi\} \tag{1}$$

If φ is a smooth and convex function, then the law is uni-valued and we get

$$\partial\varphi(\dot{\kappa}) = \{D\varphi(\dot{\kappa})\}$$

For more details on convex analysis, the reader is referred for instance to [42,48]. On this basis, the concept of potential can be extended in a weak form. We do not require more than the function φ to be convex and lower semi-continuous (with possible infinite values) and we consider multivalued laws generated by φ according to

$$\pi \in \partial\varphi(\dot{\kappa}) \tag{2}$$

The function φ is called a superpotential. The converse law takes a similar form:

$$\dot{\kappa} \in \partial\varphi^*(\pi) \tag{3}$$

where φ^* is Fenchel's transform (or conjugate) of φ

$$\varphi^*(\pi) = \sup_{\dot{\kappa}\in X}(\dot{\kappa}\cdot\pi - \varphi(\dot{\kappa}))$$

As φ is convex and lower semicontinuous, φ^* so is and $\varphi^{**} = \varphi$. Consequently, the superpotential φ and its Fenchel's conjugate φ^* satisfy Fenchel's inequality [28]

$$\forall\, \dot{\kappa}' \in X,\quad \pi' \in Y,\quad \varphi(\dot{\kappa}') + \varphi^*(\pi') \geq \dot{\kappa}'\cdot\pi' \tag{4}$$

Moreover, (2) and (3) are equivalent to

$$\varphi(\dot{\kappa}) + \varphi^*(\pi) = \dot{\kappa}\cdot\pi \tag{5}$$

For instance, the associated plasticity is obtained by taking φ^* as the indicator function I_K of the convex strength domain K (equal to zero in K and $+\infty$ otherwise), and by considering the normal flow rule (3). Nevertheless, it has been recognized experimentally that the normality rule is not relevant for many non-linear materials such as geomaterials and polymers [3,15], cyclic plasticity of metals [4], frictional contact [6] and plasticity with damage [38].

The standard way to model the non-associated plasticity is based on the use of the yield function f to define the elastic domain

$$K = \{\pi \text{ such that } f(\pi) \leq 0\}$$

and on Melan's plastic potential g to define the non-associated flow rule:

$$\exists\, \lambda \geq 0 \quad \text{such that } \dot{\kappa} = \lambda\frac{\partial g}{\partial\pi}(\pi)$$

Unfortunately, although widespread in the literature, this formalism is not a convenient framework to develop variational approaches based on functionals. The first attempt to address this challenge was the hemivariational inequality approach proposed by Panagiatopoulos [45]. Later, de Saxcé [24,21] developed the bipotential theory based on an extension of Fenchel's inequality [28], and the generalization of Moreau's superpotential [43] for the non-associated models.

By definition, a bipotential is a functional b: $X \times Y \to\ ]-\infty, +\infty]$, defined by the following properties:

(a) b is convex and lower semicontinuous in each argument.
(b) For any $\dot{\kappa}'$ and π' we have

$$b(\dot{\kappa}', \pi') \geq \dot{\kappa}'\cdot\pi' \tag{6}$$

(c) For $\dot{\kappa}$ and π we have the equivalences:

$$\pi \in \partial b(\cdot,\pi)(\dot{\kappa}) \Longleftrightarrow \dot{\kappa} \in \partial b(\dot{\kappa}, \cdot)(\pi) \Longleftrightarrow b(\dot{\kappa}, \pi) = \dot{\kappa}\cdot\pi \tag{7}$$

From a mechanical point of view, the bipotential represents the plastic dissipation power (by volume unit) and the two former conditions in (7) are the constitutive law and its inverse one. The couples $(\dot{\kappa}, \pi)$ satisfying the latter conditions in (7) are called extremal couples. If the bipotential is differentiable, the two fist relations in (7) reads

$$\dot{\kappa} = \frac{\partial b}{\partial\pi}(\dot{\kappa}, \pi),\qquad \pi = \frac{\partial b}{\partial\dot{\kappa}}(\dot{\kappa}, \pi) \tag{8}$$

Relations (7) and (8) can be qualified by implicit normality laws in the sense that (by reference to the implicit function theorems) the unknown $\dot{\kappa}$ (resp. π) belongs to both left and right hand members of the relation. For this reason, the dissipative materials admitting a bipotential are called Implicit Standard Materials. Generally speaking, these materials are non-Druckerian because the constitutive law is non-associated. This behaviour is unstable and softening may occur [7]. In the particular case of standard materials, it is easy to show that the bipotential is separated in two parts: the classical superpotential of dissipation φ and its conjugate function φ^*:

$$b(\dot{\kappa}, \pi) = \varphi(\dot{\kappa}) + \varphi^*(\pi) \tag{9}$$

Thus the fundamental inequality (6) degenerates into Fenchel's one (4).

2.2. Limit analysis of implicit standard materials

Classical theorems of limit analysis require that the structures be rigid-perfectly plastic obeying to the normal flow rule. Fortunately, the bipotential approach paves the way to variational formulation which leads to the extension of the static and kinematic theorems of limit analysis to non-associated materials [22,23,6,53,13,17] as we shall see hereafter.

Consider a rigid-perfectly plastic solid occupying the volume V with a regular boundary S. This surface S is split into two disjoint parts S_t and S_v such that $S_t \cup S_v = S$. The solid is subjected to volume forces $\overline{F}$, a surface traction distribution $\bar{t}$ on S_t and a velocity field $\bar{v}$ on S_v.

A velocity field v' is said to be kinematically admissible (K.A.) if $v' = \bar{v}$ on S_v and $d(v') = grad_s(v')$ in V. A stress field σ' is statically admissible (S.A.) if $\operatorname{div}\sigma' + \overline{F} = 0$ within V and $t(\sigma') = \sigma'\cdot n = \bar{t}$ on S_t. σ' is said to be Plastically Admissible (P.A.) if $\sigma' \in K$ where K is the