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Application of first-order canonical perturbation method with dissipative Hori-like kernel

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ABSTRACT

Lie-Hori canonical perturbation theory provides asymptotic solutions for conservative Hamiltonian systems. This restriction prevents the canonical method from being applied directly to dissipative mechanical systems. There are, however, two main alternatives to overcome this difficulty, enabling the application of canonical perturbation methods. The first one consists in constructing a time-dependent Hamiltonian, through a generating function, related to the energy dissipation pattern of the system. The second embeds the original phase space into a double dimensional one where the dynamics of the system can be formulated in a Hamiltonian way. In this paper, a modified Lie-Hori method that avoid the disadvantages of the former approaches is proposed. Namely, it is not necessary to find out a time-dependent generating function, nor doubling the number of the canonical variables of the original problem. The new algorithm provides first order analytical solutions for a certain set of dissipative non-linear dynamical systems. It is based on a suitable modification of the Hori kernel in the double-dimensional embedding phase space, allowing the inclusion of the dissipative (or generalized) forces. By means of this redefined auxiliary system, the path-integrals of the method can be performed in a domain of the phase space with the same dimensionality as the original problem.

1. Introduction

The motion of an unconstrained dynamical system with n degrees of freedom can be properly described through the Hamilton, or canonical, equations ([41], Chap. 2, Sec. 91)

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1 \ldots m. \tag{1}$$

In these $2m$ differential equations, $\mathcal{H} = \mathcal{H}(q, p, t)$ is the Hamiltonian function of the system, depending on the canonical variables p (momenta), q (coordinates), and on the time t. The canonical variables are real variables defined in a certain domain $D \subset \mathbb{R}^{2m}$, referred to as phase space, and time varies in an interval $I \subset \mathbb{R}$. $\mathcal{H}$ is assumed to be real and sufficiently regular in $D \times I$.

In many situations, i.e., for the problems named natural in Whittaker ([39], chap. III, sec. 38) terminology, $\mathcal{H}$ is the sum of the kinetic and potential energies of the system. In these cases $\mathcal{H}$ does not involve the time explicitly, and it can be identified with the total mechanical energy of the system, which is conserved in motion ([39], chap. III, sec. 41).

The analytical resolution of Eqs. (1) is not possible in general. However, many mechanical systems own a Hamiltonian function that can be split into the form

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}, \tag{2}$$

with $|\Delta\mathcal{H}| \ll |\mathcal{H}_0|$, i.e., $\Delta\mathcal{H}$ is a perturbation of $\mathcal{H}_0$, usually referred to as unperturbed Hamiltonian. If the dynamics generated by $\mathcal{H}_0$ is known and some additional conditions hold ([2], chap. 10), an asymptotic solution of the dynamics corresponding to $\mathcal{H}$ can be obtained with the aid of canonical perturbation theories.

The development of canonical perturbation theories[1] began in the second half of the 19th century. Such theories were mainly concerned with the resolution of some important problems of Celestial Mechanics like, for example, the lunar theory [8]. Basically, the idea of the method consists in determining a canonical transformation built from a certain function (*determining* or *generating* function), which leads to canonical equations easier to integrate.

Subsequent researches pushed those theories forward, specially with the works by Poincaré [33] and von Zeipel [43]. The last method played an important role in the determination of the motion of an

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[1] For a comprehensive treatment of canonical perturbation theories we refer the reader, for example, to Nayfeh [32] and Ferraz-Mello [14].

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artificial satellite ([5], chap. XVII, sec. 12).

A main achievement was due to Hori [23], who introduced a perturbation method based on Lie series, allowing a simpler handling of canonical transformations. It is often referred to as Lie-Hori canonical method. Later, a close approach was designed by Deprit [9], both theories being equivalent[2] [6]. Lie-Hori's method presents some advantages [6] with respect to that of von Zeipel's [43]. Specifically: the determining function of the transformation just depends on the transformed canonical variables; the theory is formulated through Poisson brackets,[3] hence it is canonically invariant; and it is possible to provide the expression of any function of the initial canonical set in terms of the transformed one.

In its original formulation, Hori's method can just be applied to Hamiltonians independent of time, i.e., $\mathcal{H} = \mathcal{H}(q, p)$. Even so, this restriction can be easily circumvallated by introducing the extended phase space of dimension $2m + 2$. With this construction, also known as homogeneous formalism, the time assumes the role of a new canonical coordinate with conjugated momentum given by $-\mathcal{H}$ ([41], chap. 2, sec. 93, [36], i.a., see Section 2).

In contrast, the application of Hori's method to dynamical systems affected by dissipative processes (for example, damped harmonic oscillators) cannot be carried out in a simple way. This is due to the fact that the construction of the generating function implies the existence of a privileged dynamical system related to the unperturbed system, called auxiliary system or *Hori kernel* ([14], chap. 6, sec. 6.5), which has the restriction of being Hamiltonian. Therefore, the generalized canonical systems, which are characterized by the differential equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} - Q_{p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + Q_{q_i}, \ i = 1\ldots m, \tag{3}$$

cannot be included in this category (hereafter, time-derivative will be denoted by a dot). In these equations, Q_{p_i} and Q_{q_i} are the generalized (or canonical) forces, whose inclusion is necessary to account for the dissipation of the system. That kind of equations appears, e.g., when treating the drag action on the motion of an artificial satellite [5]. When $\mathcal{H} = 0$, Eqs. (3) reduce to the most general form of a first order differential system with even unknowns [36]. Indistinctly, generalized canonical systems will be also denominated as non-Hamiltonian ones.

For the obtention of an asymptotic solution of Eq. (3) when viewed as a perturbation of $\mathcal{H}_0$ and the zeroth-order part of the generalized forces, there exist specific perturbation algorithms like those based on the method of averaging [4] or on an extension of the Lie series methods (e.g., Kamel [25], Henrard [21], Hori [24]).

Nevertheless, it is still possible to use the original Hori's method with proper modifications of Eqs. (3). Basically, two different ways can be followed to accomplish this procedure.

On the one hand, it is possible to find a time-dependent canonical transformation in order to obtain the Eqs. (3) from the Hamiltonian of the associated non-dissipative dynamical system, i.e., with no generalized forces. Since the canonical transformation depends on time, it will also be the case for the transformed Hamiltonian. However, it does not pose any obstacle, since the problem can be formulated in the extended phase space where the Lie-Hori method can be applied. A major difficulty of this approach is that there is no systematic way to find that canonical transformation, with the exception of some simplified dynamical systems like harmonic oscillators (e.g., Nagem et al. [31]). For them, it is possible to have some *a priori* knowledge about the energy dissipation features in the system evolution. It makes feasible to construct the successful canonical transformation from the non-dissipative dynamical system to recover the original dissipative dynamics (see Section 5.1).

The second possibility is to hamiltonize the equations of motion by constructing a single Hamiltonian $\widetilde{\mathcal{H}}$, necessarily different from $\mathcal{H}$, in order to derive Eqs. (3). This approach is originally attributed to Liouville, and it is already considered in Birkhoff [3].

Within this category, a general procedure consists in embedding the original $2m$-dimensional system into a $4m$-dimensional phase space (or $4m + 4$ in the explicitly time-dependent case), and determine the new Hamiltonian $\widetilde{\mathcal{H}}$. In the context of perturbations theories this procedure can be found, for example, in Kamel [26], Hori [24], and specially in Choi and Tapley [7], where Hori's original algorithm is utilized once the embedding procedure has been applied. Although the application of canonical perturbation theories in this approach is straightforward from an analytical point of view, the management of the double number of canonical variables is involved in practice and become a main disadvantage of the procedure.

This research focuses on a certain set of dissipative dynamical systems whose analytical asymptotic solution of first-order can be obtained from Hori's method, without the need of doubling the dimension of the phase space. In this way, the disadvantages of the former procedures for general dynamical systems can be avoided, while preserving their benefits.

Those particular dynamical systems are characterized by the fact that their unperturbed part, which must include canonical forces, gives rise to a linear system of differential equations with constant coefficients with respect to the $2n$ canonical variables p_i and q_i, $i \leq n$. This condition is not really restrictive in practice, since any unperturbed Hamiltonian that is integrable (in Liouville sense) can be expressed in angle-action variables, which produce linear equations of motion. Of course, the form of linear system is attainable in different manners.

The system may include $2(m - n)$ additional canonical variables p_j and q_j, $n < j \leq m$, which do not enter into the unperturbed dynamics, i.e., they are non-coupled variables (solved independently from the $2n$ preceding ones), or even cyclic or ignorable variables.[4] The perturbation stems from a non-linear Hamiltonian $\Delta\mathcal{H} = \mathcal{H}_1$, which is a function of the whole canonical set and time of the form

$$\begin{aligned}\mathcal{H}_1 = &\sum_{i=1}^{n} [f_{p_i}(q_{n+1}, \ldots q_m, p_{n+1}, \ldots p_m, t) p_i + f_{q_i}(q_{n+1}, \ldots q_m, p_{n+1}, \ldots p_m, t) q_i] \\ &+ f_t(q_{n+1}, \ldots q_m, p_{n+1}, \ldots p_m, t),\end{aligned} \tag{4}$$

f_{p_i}, f_{q_i} and f_t being real and sufficiently regular functions, but otherwise, arbitrary. A remarkable example of such kind of perturbations appears in the Hamiltonian theory of the rotation of a two-layer non-rigid Earth, e.g., Getino and Ferrándiz [16–18], as will be studied as an application of the method in Section 6.

The paper is structured as follows. In Section 2, the main features of the first-order Lie-Hori canonical method and the homogeneous formalism are exposed. In the subsequent Sections 3 and 4, the proposed modification to the perturbation method is developed constructively, including some important mathematical properties. This comprises the definition of the extended dynamical system within the double dimensional phase space, and the particular study of the previously stated non-Hamiltonian systems, allowing the reduction of the dimensionality of the problem. As it will be shown, the procedure is based on a suitable definition of an Hori-like kernel of the perturbation method. In Section 5, different approaches to tackle a dissipative system

[2] Although there are slight differences in the approach of both methods ([30], app. C and D), they are sometimes referred to as Lie-Hori, Lie-Deprit, or even Hori-Deprit method, indistinctly.

[3] The Poisson bracket of two smooth functions f and g of the canonical set is defined by the bilinear operation

$$\{f(q, p), g(q, p)\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right).$$

[4] In the latter case, cyclic variables are considered with respect to the unperturbed Hamiltonian. Then, the constant coefficients of the linear system can depend on the conjugated momenta of these cyclic variables.