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The use of differential and non-local transformations for numerical integration of non-linear blow-up problems

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ABSTRACT

Two new methods of numerical integration of Cauchy problems for nonlinear ODEs of the first- and second-order, which have blow-up solutions are described. In such problems, the position of the singular point is not known in advance. The first method is based on obtaining an equivalent system of equations by applying a differential transformation, where the first derivative (given in the original equation) is chosen as a new independent variable, $t = y'_x$. The second method is based on introducing a new auxiliary non-local variable of the form $\xi = \int_{x_0}^{x} g(x, y, y'_x)\,dx$ with the subsequent transformation to the Cauchy problem for the corresponding system of coupled ODEs. Both methods lead to problems whose solutions are represented in parametric form and do not have blowing-up singular points; therefore the standard fixed-step numerical methods can be applied. The efficiency of the proposed methods is illustrated with a number of test problems that admit exact solutions. It is shown that the methods, based on special exp-type transformations (which are particular cases of the general non-local transformation), are more efficient than the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation. The method, based on introducing a non-local variable, can be generalized to the nth-order ODEs and systems of coupled ODEs.

1. Introduction

We will consider Cauchy problems for ODEs, whose solutions tend to infinity at some finite value of x, say $x = x_*$. Such x_* does not appear explicitly in the given differential equation and it is not known in advance. Similar solutions exist on a bounded interval (hereinafter in this article we assume that $x_0 \le x < x_*$) and are called blow up solutions. This raises the important question for practice: how to determine the position of a singular point x_* and the solution in its neighborhood with the aid of numerical methods.

In general, the blow-up solutions, that have a power singularity, can be represented in a neighborhood of the singular point x_* as

$$y \simeq A(x_* - x)^{-\beta}, \quad \beta > 0, \tag{1}$$

where A is a constant. For these solutions we have $\lim_{x\to x_*} |y| = \infty$ and $\lim_{x\to x_*} |y'_x| = \infty$.

For blow-up solutions with the power singularity (1) near the singular point x_* we have

$$y'_x/y \simeq \beta/(x_* - x), \tag{2}$$

i.e. the required function grows more slowly than its derivative. Therefore, we have $\lim_{x\to x_*} y'_x/y = \infty$ (this is a common property of any blow-up solutions; it must be taken into account when carrying out numerical calculations).

The direct application of the standard fixed-step numerical methods in such problems leads to certain difficulties because of the singularity in the blow-up solutions and the unknown (in advance) range of variation of the independent variable x (see, for example, [1,2]).

One of the basic ideas of numerical solution of such problems is to find a suitable transformation, leading to the equivalent problem for one differential equation or a system of coupled equations, the solutions of which have no singularities at unknown point.

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Currently, two methods based on this idea are most commonly used. The first one is proposed by Acosta et al. [3]. They have suggested to apply a hodograph transformation $x = \bar{y}$, $y = \bar{x}$, where the independent and dependent variables are reversed. The second method, which is based on the arc-length transformation, has been proposed by Moriguti et al. [4] (for details, see below Items 2° in Sections 3.1 and 5.1 as well as Ref. [5]). This method is rather general and it can be applied for numerical integration of systems of ODEs.

The methods based on the hodograph and arc-length transformations for blow-up solutions with a power singularity of the form (1) lead to the Cauchy problems whose solutions tend to the asymptote with respect to the power law for large values of the new independent variable. This creates certain difficulties in some problems, since one has to consider large intervals of variation of the independent variable in numerical integration.

Based on other ideas, some special methods of numerical integration of blow-up problems are described, for example, in [1,2,5–9].

In this paper, we propose two new methods of numerical integration of non-linear Cauchy problems for ODEs of the first- and second-orders, which have blow-up solutions. These methods are based on the differential and non-local transformations allowing us to obtain the equivalent systems of ODEs, whose solutions do not have singularities at some a priori unknown point. It is shown that special exp-type transformations (which are particular cases of the general non-local transformation) lead to the Cauchy problems whose solutions tend exponentially to the asymptote (which determines the position of the required singular point x_*) for large values of the new independent variable; therefore exp-type transformations are more preferable than the hodograph and arc-length transformations.

2. Problems for first-order equations. Differential transformations

2.1. Solution method based on a differential transformation

The Cauchy problem for the first-order differential equation has the form

$$y'_x = f(x, y) \quad (x > x_0), \tag{3}$$

$$y(x_0) = y_0. \tag{4}$$

In what follows we assume that $f = f(x, y) > 0$, $x_0 \geq 0$, $y_0 > 0$, and $f/y^{1+\varepsilon} \to \infty$ as $y \to \infty$, where $\varepsilon > 0$ (in such problems, blow-up solutions arise when the right-hand side of a non-linear ODE is quite rapidly growing as $y \to \infty$).

First, we present the ODE (3) as an equivalent system of differential-algebraic equations

$$t = f(x, y), \qquad y'_x = t, \tag{5}$$

where $y = y(x)$ and $t = t(x)$ are unknown functions to be determined.

By applying (5), we derive a system of equations of the standard form, assuming that $y = y(t)$ and $x = x(t)$. By taking the full differential of the first equation in (5) and multiplying the second one by dx, we get

$$dt = f_x\,dx + f_y\,dy, \quad dy = t\,dx, \tag{6}$$

where f_x and f_y are the respective partial derivatives of $f = f(x, y)$. Eliminating first dy, and then dx from (6), we obtain a system of the first-order coupled equations

$$x'_t = \frac{1}{f_x + tf_y}, \qquad y'_t = \frac{t}{f_x + tf_y} \quad (t > t_0), \tag{7}$$

which must be supplemented by the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad t_0 = f(x_0, y_0). \tag{8}$$

Conditions (8) are derived from (4) and the first equation of (5).

Assuming that the conditions $f_x + tf_y > 0$ at $t_0 < t < \infty$ are valid, the Cauchy problem (7)–(8) can be integrated numerically, for example, by applying the Runge–Kutta method or other standard numerical methods (see for example [10–15]). In this case, the difficulties (described in the introduction) will not occur since x'_t rapidly tends to zero as $t \to \infty$. The required critical value x_* is determined by the asymptotic behavior of the function $x = x(t)$ for large t.

2.2. Test problems. Exact and numerical solutions

Let us illustrate the method proposed in Section 2.1 with simple examples.

Example 1. Consider the model Cauchy problem for the first-order ODE with separated variables

$$y'_x = y^2 \quad (x > 0), \qquad y(0) = a, \tag{9}$$

where $a > 0$. The exact solution of this problem has the form

$$y = \frac{a}{1 - ax}. \tag{10}$$

It has a power-type singularity (a first-order pole) at a point $x_* = 1/a$ and does not exist for $x > x_*$.

By introducing a new variable $t = y'_x$ in (9), we obtain the following Cauchy problem for the system of equations:

$$\begin{aligned} &x'_t = \frac{1}{2ty}, \qquad y'_t = \frac{1}{2y} \quad (t > t_0);\\ &x(t_0) = 0, \qquad y(t_0) = a, \quad t_0 = a^2, \end{aligned} \tag{11}$$

which is a particular case of the problem (7)–(8) with $f = y^2$, $x_0 = 0$, and $y_0 = a$. The exact solution of this problem has the form

$$x = \frac{1}{a} - \frac{1}{\sqrt{t}}, \qquad y = \sqrt{t} \quad (t \geq a^2). \tag{12}$$

It has no singularities; the function $x = x(t)$ increases monotonically for $t > a^2$, tending to the desired limit value $x_* = \lim_{t\to\infty} x(t) = 1/a$, and the function $y = y(t)$ monotonously increases with increasing t. The solution (12) for the system (11) is a solution of the original problem (9) in parametric form.

The maximum error of the numerical solution of the Cauchy problem for system of Eqs. (11) with $a = 1$ obtained by the classical Runge–Kutta method of the fourth-order approximation for stepsize $h = 0.2$ does not exceed 0.017% for $y \leq 50$.

Remark 1. Here and in what follows, the numerical integration interval for the new variable t (or ξ) is usually determined, for demonstration calculations, from the condition $\Lambda_{\rm m} = 50$, where

$$\Lambda_{\rm m} = \min\{|y|,\ y'_x/y\} \quad (\text{for } |y_0| \sim 1 \text{ and } |y_1| \sim 1), \tag{13}$$

and $y_1 = y'_x(x_0)$. In a few cases, the condition $\Lambda_{\rm m} = 100$ or $\Lambda_{\rm m} = 150$ is used, which is specially stipulated. In the definition of $\Lambda_{\rm m}$, a relation y'_x/y is included that takes into account the property (2). For first-order ODE problems of the form (3)–(4), the definition of $\Lambda_{\rm m}$ can be replaced by the equivalent definition $\Lambda_{\rm m} = \min\{|y|,\ f/y\}$.

Conditions $|y_0| \sim 1$ and $|y_1| \sim 1$ in (13) are not strongly essential, since the substitution $y = y_0 - 1 + (y_1 - 1)(x - x_0) + \bar{y}$ leads to an equivalent problem with the initial conditions $\bar{y}(x_0) = \bar{y}'_x(x_0) = 1$.

Example 2. For a more general two-parameter Cauchy problem,

$$y'_x = y^\gamma, \qquad y(0) = a > 0,$$

having a blow-up solution for $\gamma > 1$, the introduction of a new variable $t = y'_x$ leads to the system of equations of the form (7), the solution of which is determined by the formulas

$$x = \frac{1}{\gamma - 1}\left(a^{1-\gamma} - t^{\frac{1-\gamma}{\gamma}}\right), \qquad y = t^{\frac{1}{\gamma}} \quad (t \geq a^\gamma). \tag{14}$$

This solution behaves qualitatively similar to the solution (12) as $t \to \infty$.