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Effects of voltage change on the dynamics in a comb-drive finger of an electrostatic actuator

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A B S T R A C T

Here we consider a transverse comb-drive device with one moveable finger between two fixed ones when a harmonic load driven DC–AC voltage $V_\delta(t) = v_0 + \delta v(t)$, $v(t) \in C(\mathbb{R}/T\mathbb{Z})$, v_0-constant and $\delta \in [0, \Delta_c]$ is added for some appropriate Δ_c. By means of the Leray–Schauder Continuation Theorem we find families of symmetric periodic solutions emanating from the DC-voltage case ($\delta = 0$) for all positive values of the parameter δ, providing qualitative and quantitative information of this families for a computable Δ_c avoiding collision of the fingers.

1. Introduction

In this document we consider a special type of micro-electro-mechanical-system (MEMS) known in the literature as *Comb-drive fingers*. These devices are found in a wide range of applications, mainly for sensing and actuation, such as resonant sensors [1], accelerometers [2] and optical communication devices [3]. An excellent bibliographic source of applications of this devices can be found in [4] and the references therein. The precise configuration of the comb-drive finger that we consider here is illustrated in Fig. 1, where a moveable electrode (*finger*) is sandwiched between two stationary electrodes and moves in the transverse direction to the longitudinal axis of the stationary electrodes. The moveable finger with mass m, is attached to a linear spring with stiffness coefficient $k > 0$, and it is at the center of the two stationary electrodes at a distance d and it is biased from the upper and lower sides both with the same voltage source, with V load. Under this configuration, two electrostatic forces act on the finger, one from the upper side $\mathbf{F}_{\text{upper}}$ in the negative x direction an the other on its lower side $\mathbf{F}_{\text{lower}}$ in the positive x direction. The magnitudes of these attractive forces are given by[1]

$$F_{\text{lower}} = \frac{eL\varepsilon V^2}{2(d-x)^2}, \quad F_{\text{upper}} = \frac{eL\varepsilon V^2}{2(d+x)^2},$$

where x is the vertical displacement of the finger ($|x|$ is always assumed to be less than d), ε is the absolute dielectric constant of vacuum, e is the width of the electrodes and L is the long part of the finger. By the second Newton's law the equation of motion of the finger is given by the following second order nonlinear differential equation

$$m\ddot{x} + kx = \left(\frac{1}{(d-x)^2} - \frac{1}{(d+x)^2}\right)\frac{eL\,\varepsilon\,V^2}{2}, \tag{1}$$

Along this document, we consider a DC–AC voltage V of the form

$$V = V(t,\delta) = v_0 + \delta v(t), \tag{2}$$

with v_0 as the DC load (a positive constant) and $\delta v(t)$ as the AC load, where $v(t) \in C(\mathbb{R}/T\,\mathbb{Z})$ with zero average and $\delta \in [0, \Delta[$ (a control parameter). The Eq. (1) can be written in the form

$$\ddot{x} + \omega^2 x = \frac{4dh\,xV^2(t,\delta)}{(d^2-x^2)^2}, \tag{3}$$

where $\omega^2 = k/m$, $h = \frac{eL\varepsilon}{2m} > 0$.

A matter of practical and theoretical interest is to control the DC–AC voltage values in order to avoid the collision between the movable finger and the fixed ones, an undesirable effect known as *pull-in instability* that could damage the device. Since the pioneering works of Nathanson et al. [5] and Taylor [6] on the pull-in phenomenon in the late 1960 many authors have focused on characterizing and

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[1] The reader can find the deduction of this electrostatic forces between the electrodes, in terms of the capacitance of the parallel-plate capacitors and the voltage load in [4, chapter 3, section 5].

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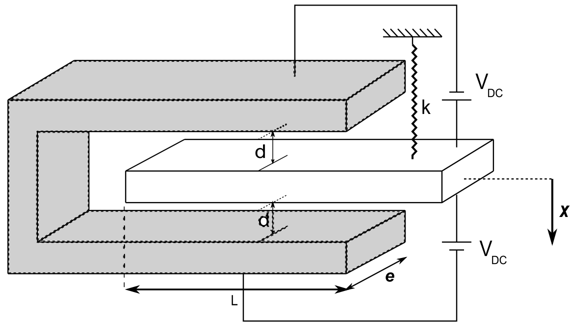

Fig. 1. Idealized mass–spring model of a *Comb-Driver finger* of electrostatic actuator.

understanding it in many different directions, both analytically and numerically. Some of this mathematical studies are cited on [7–13]. A more complete and recent survey of the literature can be found in [14]. All these contributions show the importance of analytical and numerical studies of possible oscillatory motions of (3) that avoid the difficulties associated with the electrostatic nonlinearity and the pull-in phenomenon. As far as we known, the first result using topological and variational methods in order to study the existence and stability of periodic oscillations in canonical MEMS like [5], were given in [15]. Motivated for this type of mathematical techniques, we study here the existence of even and nT-periodic solutions (n positive integer) for the MEMS (1) in a quantifiable δ-interval. In our approach, we follow some of the main ideas developed in [16,17]. A remarkable difference respect to the model studied in [16,17] is the presence of two singularities in the variable x and as consequence, the technical difficulties arising in order to guarantee that the solutions are well defined in $[0, nT]$ for an adequate region of initial conditions and parameters (see Section 3). The main objective of this paper is to study analytically these families of symmetric periodic orbits of the movable finger for positive values of the parameter δ, proving that some periodic orbits for $\delta = 0$ can be continue to all value of $\delta \in [0, \Delta_c]$ for a computable Δ_c. The main tool for proving our result is the global continuation method of the zeros of a function depending on one parameter provided by the Leray and Schauder Theorem and based in the Brouwer degree [16,18,19]; see Section 3. Our study is organized as follows. In Section 2, we collect some known results of the autonomous case ($\delta = 0$) that will be useful later. In Section 3, we reduce the search of even periodic solution of (3) to study the set of zeros of an appropriate function in order to use an analytical version of the Global Continuation Theorem of Leray–Schauder. Also, we show in the Proposition 3 sufficient conditions for the prolongability of solutions (3) up to the half period. The proof of this technical result will be made in the Appendix, using a lemma of estimation for growing of oscillations for second order equation developed in [20]. In Section 4, we present the statements and proofs of the main results in this document, Theorems 1–3. Finally, we present some numerical examples of the theoretical results obtained in this work.

2. The autonomous case

In this section we study the dynamics of (3) when the voltage source is constant, i.e. $V(t) = v_0$, with v_0 a positive constant. In this case the Eq. (3) becomes the autonomous equation

$$\ddot{x} + \omega^2 x = \frac{4v_0^2 d h x}{(d^2 - x^2)^2}. \tag{4}$$

The above equation defines an integrable Hamiltonian system of one degree of freedom with Hamiltonian

$$\mathcal{H}(x, \dot{x}) = \frac{\dot{x}^2}{2} + \frac{x^2}{2}\left(\omega^2 - \frac{4v_0^2 h}{d(d^2 - x^2)}\right). \tag{5}$$

Although $\mathcal{H}$ has five the critical points, two of them are at a distance greater than d and therefore do not have physical meaning in the model. A straightforward computation provides a threshold value for the voltage load

$$v^* = \frac{\omega}{2}\sqrt{\frac{d^3}{h}}, \tag{6}$$

such that if $v_0 < v^*$ the critical points of (5) are $(x_*, 0), (0, 0), (-x_*, 0)$, with $|x_*| < d$ where

$$x_* = \sqrt{d^2 - \frac{2}{\omega}\sqrt{d v_0^2 h}}. \tag{7}$$

As a direct consequence of the previous discussions we have the following lemma.

Lemma 1. *If* $0 < v_0 < v^*$ *the* Eq. (4) *has three trivial solutions (equilibria) in* $]-d, d[$ *given by*

$(0, 0), \quad (x_*, 0), \quad (-x_*, 0).$

Moreover, the origin is a local minimum (center) of $\mathcal{H}$ *and the other two equilibria are relative maxima (saddles) of* $\mathcal{H}$.

Remark 1. It is worth to mention that when $v_0 > v^*$ the critical point of $\mathcal{H}$ in $]-d, d[$ is the origin which is a saddle point, physically this means that the moveable finger gets stuck to one of the other fixed electrodes, this effect is known in the comb-drive literature as *pull-in of double-sided* capacitors or *side instability*.

Since the orbits of any solution lie on the energy levels $\mathcal{H}(x, \dot{x}) = c$; c-constant. From Lemma 1 we deduce that $c \in\ ]-\infty, \mathcal{H}_*]$ with $\mathcal{H}_* := \mathcal{H}(x_*, 0)$ and depending of the value of c we obtain different types of orbits on the phase plane $(x, \dot{x})$, see Fig. 2. When $c = 0$ we have the trivial solution $x(t) \equiv 0$, which corresponds to the stationary position of the finger and if $0 < c < \mathcal{H}_*$ we have closed orbits corresponding to periodic orbits.

Consider the connected set $U = \{(x, \dot{x}) : 0 < \mathcal{H}(x, \dot{x}) < \mathcal{H}_*\}$ which contains the line segment $S_0 = \{(\xi, 0) : 0 < \xi < x_*\}$. By the symmetry of $\mathcal{H}$, for each $\xi \in S_0$ the periodic solution $x(t, \xi)$ of (4) that satisfy the initial conditions

$$x(0) = \xi, \quad \dot{x}(0) = 0, \tag{8}$$

crosses S_0 only one time. Then, there is a minimal return time $\mathcal{T}(\xi)$ where

$x(t + \mathcal{T}(\xi), \xi) = x(t, \xi),$

for all $t \in \mathbb{R}$. From now on $\mathcal{T}(\xi)$ will be the minimal period of $x(t, \xi)$.

Proposition 1. *The period function* $\mathcal{T}(\xi)$ *satisfies the following properties*

(a) $\frac{d\mathcal{T}(\xi)}{d\xi} > 0$ *for all* $0 < \xi < x_*$,
(b) $\lim_{\xi \to 0^+} \mathcal{T}(\xi) = \frac{2\pi}{\sqrt{\omega^2 - 4v_0^2 h/d^3}}$,
(c) $\lim_{\xi \to x_*^-} \mathcal{T}(\xi) = \infty$.

The time $\mathcal{T}_L := \frac{2\pi}{\sqrt{\omega^2 - 4v_0^2 h/d^3}}$ *is the period of the linearized system of* (4) *in the origin.*

Proof. Let $x(t, \xi)$ be a periodic solution of (4) that satisfy the initial conditions (8). Consider the level set $\{(x, \dot{x}) : \mathcal{H}(x, \dot{x}) = \mathcal{H}(\xi, 0)\}$ with

$$\mathcal{H}(\xi, 0) = \frac{\xi^2}{2}\left(\omega^2 - \frac{4v_0^2 h}{d(d^2 - \xi^2)}\right).$$

By the symmetries of $\mathcal{H}$ and (7) a straightforward computations show that (See Fig. 3)

$$\mathcal{T}(\xi) = \frac{2}{\sqrt{v_0^2 h d}} \int_0^\xi \left[\left(\frac{1}{(d^2 - (x_*)^2)^2} - \frac{1}{(d^2 - \xi^2)(d^2 - x^2)}\right)(\xi^2 - x^2)\right]^{-1/2} dx.$$