International Journal of Non-Linear Mechanics 93 (2017) 47–52

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

On the near-critical behavior of cavitation in elastic plane membranes

Y.N. Geng[a], Z.X. Cai[a,*], Y.B. Fu[a,b,**]

[a] Department of Mechanics, Tianjin University, Tianjin 300072, China
[b] Department of Mathematics, Keele University, Staffordshire ST5 5BG, UK

ARTICLE INFO

Keywords:
Cavitation
Imperfection sensitivity
Void growth
Membrane

ABSTRACT

Material cavitation under tensile loading is often studied by assuming the pre-existence of a small void. In this case the void would initially grow but without significant change in its size, and cavitation is said to take place if this slow growth is followed by rapid growth at higher load values. In the limit when the original void radius δ tends to zero, there will be no growth until a load or stretch measure, λ say, reaches a well-defined critical value λ_{cr} at which a cavity appears suddenly. In this paper we study the near-critical asymptotic behavior of cavitation in plane membranes when δ is not zero but small, and show that the near-critical behavior is governed by a scaling law in the form $\lambda - \lambda_{cr} = C(\delta/L)^m$, where L is the undeformed outer radius of the plane membrane, and C and m are non-dimensional constants. The positive power m in general depends on the material model used, but for the three classes of material models considered, it happens to be equal to $2(1+\nu)/(3+\nu)$ in each case, where ν is Poisson's ratio for infinitesimal deformations. If a pre-existing void is viewed as an imperfection, then this scaling law describes the imperfection sensitivity of cavitation: it states that in the presence of imperfections significant void growth would occur if λ were increased to within an order $(\delta/L)^m$ interval around λ_{cr}.

1. Introduction

In some tension experiments on rubber materials [1–4], it is shown that internal micro-voids could nucleate suddenly (known as cavitation) under certain critical loading conditions. With continued loading a series of cavities may grow, coalesce, and eventually form large enclosed cracks. Thus, cavitation may signal the onset of material failure.

Cavitation in rubber materials was first demonstrated experimentally by Gent and Lindley [1] who also provided a theoretical explanation, but it was not until after Ball [5] had formulated it as a rigorous bifurcation problem that an explosive growth of interest followed. In its simplest form, namely an isotropic sphere with a small void at its center that is subjected to a hydrostatic tension on its outer surface, the cavitation problem consists of solving a two-point boundary value problem involving a single nonlinear second-order ordinary differential equation. The critical tension can be obtained by taking two limits one after the other: firstly the undeformed cavity radius tending to zero, and then the deformed cavity radius tending to zero. This has been demonstrated by Horgan and Abeyaratne [6], and is justified by the rigorous results of Sivaloganathan et al. [7]. When the material is incompressible, the radially axisymmetric deformation can be determined to within an arbitrary constant irrespective of the form of the strain-energy function, and determination of the critical tension and post-buckling deformation is then reduced to the evaluation of an integral. When the material is compressible, the two-point boundary value problem can be solved by a shooting procedure in the most general case (see, e.g., [8]), but many studies have focused on finding closed-form solutions for specific material models (see, e.g., [9–12]). There also exists a large body of literature concerned with the effects of anisotropy, material inhomogeneity, surface tension, and plastic behavior; see, e.g., [13–19]. We refer to [20,21] for a comprehensive review of the literature. An interesting result that deserves special mentioning is that whereas cavitation in a homogeneous isotropic solid sphere is a supercritical bifurcation, cavitation may change into a subcritical bifurcation when material inhomogeneity or anisotropy is taken into account.

Because of its relevance in a wide range of applications, cavitation in solids is still a topic of active research. Cristiano et al. [22] and Hamdi et al. [23] have recently conducted further experiments in order to understand the connection between cavitation and material fracture. Dorfmann and Burtscher [24] and Kumar et al. [25] suggested that cavitation was the primary cause of irreversible stress softening in seismic bearings; the same opinion was expressed for metal materials [26–28]. Increased attention has also been paid to cavitation associated with general geometries and loading conditions, and its numerical

* Corresponding author at: Department of Mechanics, Tianjin University, Tianjin 300072, China
** Corresponding author at: Department of Mathematics, Keele University, Staffordshire ST5 5BG, UK.
E-mail addresses: zxcai@tju.edu.cn (Z.X. Cai), y.fu@keele.ac.uk (Y.B. Fu).

http://dx.doi.org/10.1016/j.ijnonlinmec.2017.04.012
Received 7 April 2017; Accepted 12 April 2017
Available online 15 April 2017

computations [29–33]. More closely related to the current study is the growing interest in cavitation in soft and biological materials. For example, David and Humphrey [34] studied the stress and strain concentration due to the introduction of a circular hole in an anisotropic bio-tissue, Merodio and Saccomandi [35] studied the effect of fibre-reinforcement in the radial direction, McMahon et al. [36] and Pence and Tsai [37,38] investigated cavitation due to growth and swelling, respectively, and Volokh [39] suggested that cavitation instability could be a rational indicator of aneurysm rupture.

This paper is concerned with the asymptotic properties of cavitation solutions, an aspect of the cavitation problem that seems to have been under-studied. Horgan and Abeyaratne [6] considered cavitation associated with a Blatz-Ko material, and derived two asymptotic expressions for the deformed void radius $r(\delta)$ valid for $\lambda < \lambda_{cr}$ and $\lambda > \lambda_{cr}$, respectively, where δ is the undeformed void radius, λ is the azimuthal stretch imposed at the outer surface and λ_{cr} its critical value. It was shown that when $\lambda < \lambda_{cr}$, the deformed radius $r(\delta)$ is of order δ, whereas when $\lambda > \lambda_{cr}$, the leading order term in $r(\delta)$ is independent of δ, indicating that in this parameter regime the azimuthal stretch $r(\delta)/\delta$ would tend to infinity as $\delta \to 0$. We note, however, that these expansions would break down in the limit $\lambda \to \lambda_{cr}$, which is the parameter regime to be examined in the current paper. Some asymptotic results have been derived by Negrón-Marrero and Sivaloganathan [40] to aid their numerical calculations. In particular, they showed that even for a general strain-energy function the deformed cavity radius is given by $r(0) = C(\lambda - \lambda_{cr})^{1/n}$ to leading order as λ approaches λ_{cr}, where C is a positive constant and n is the dimension of the cavitation problem.

It is well-known that although cavitation is a bifurcation phenomenon, it cannot be studied using the traditional methods of linear and weakly nonlinear bifurcation analysis – it is an intrinsically nonlinear problem. Despite this difference, the bifurcation diagram, showing the dependence of cavitation size on the applied tensile pressure, is nonetheless of the same shape as that for more traditional buckling problems such as the pitch-fork bifurcation associated with Euler buckling. A pre-existing void can be viewed as an imperfection, and the associated imperfect bifurcation diagram can be viewed as an "unfolding" of the perfect bifurcation diagram, in exactly the same manner as in Euler buckling; see, e.g., Fig. 1 in [6], or the three figures in the current paper. In the case of Euler buckling, a simple near-critical analysis would yield an amplitude equation of the form

$$(p - p_{cr})A + c_1 A^3 + c_2 \delta = 0, \tag{1}$$

where A is a measure of the *unscaled* buckling amplitude (e.g. the maximum deflection), p is the compressive load and p_{cr} its critical value, c_1 and c_2 are constants, and δ denotes the amplitude of the imperfection. According to this amplitude equation, when p is much smaller than p_{cr} so that $p - p_{cr}$ is finite, dominant balance is between the first and third terms in (1), which yields $A \sim \delta$, where "$\sim$" means "is of the same order as". On the other hand, when $p - p_{cr}$ is sufficiently small so that all the three terms in (1) are of the same order, we have

$$A \sim \delta^{1/3}, \quad \text{and} \quad p - p_{cr} \sim \delta^{2/3},$$

which are the scalings of most interest in the assessment of structural integrity. The final parameter regime of interest is when $p - p_{cr}$ is much larger than $\delta^{2/3}$, in which case dominant balance is between the first two terms, the effect of imperfection is not felt to leading order, and so the bifurcation curve tends to its counterpart in the absence of imperfections.

Thus, an amplitude equation such as (1) serves to capture the near-critical behavior, and it is well-known that the solution given by (1) gives a very good approximation to the exact solution around $p = p_{cr}$ even when δ is only moderately small. The main objective of the current study is to demonstrate that a near-critical amplitude equation analogous to Eq. (1) can also be derived for the cavitation problem. We shall consider three material models for which the cavitation solution can be obtained in closed-form.

The rest of this paper is organized as follows. After formulating the cavitation problem in the next section, we present the above-mentioned asymptotic analysis for three classes of material models in the subsequent sections. In the final section we reflect on our main results and conclude the paper with some additional comments.

2. Problem description

We consider a uniform circular membrane containing a pre-existing circular hole with radius δ at its center, and the membrane is subjected to radial tension. The undeformed configuration occupies the region $D_0 = \{(R, \Theta) | \delta \le R \le 1, 0 \le \Theta \le 2\pi\}$ in terms of plane polar coordinates, where the outer radius of the membrane has been taken to be unity, corresponding to the fact that we are using the actual radius as the unit of lengths. We assume that the resulting axisymmetric deformation is given by

$$r = r(R), \quad \theta = \Theta, \tag{2}$$

where r and θ are the plane polar coordinates in the current configuration, and the function $r(R)$ is to be determined. The associated principal stretches of the deformation are then given by

$$\lambda_r = \frac{dr}{dR}, \quad \lambda_\theta = \frac{r}{R}. \tag{3}$$

In the subsequent analysis, the subscripts r and θ are interchangeable with 1 and 2, respectively. We shall denote the azimuthal stretch at the outer boundary $R=1$ by λ, that is $r(1) = \lambda$, and take λ as the control parameter in the subsequent bifurcation analysis. The deformed configuration then occupies the region $D = \{(r, \theta) | r(\delta) \le r \le \lambda, 0 \le \theta \le 2\pi\}$.

In the absence of body forces, the only equilibrium equation that is not satisfied automatically can be written as

$$\frac{d\sigma_r}{dR} + \frac{dr}{dR}\frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{4}$$

where σ_r and σ_θ are the principal Cauchy stress components (measured per unit length in the deformed configuration). When the membrane is viewed as a two-dimensional elastic continuum with strain energy function $W(\lambda_r, \lambda_\theta)$ (measured per unit area in the undeformed configuration), the Cauchy stresses are given by

$$\sigma_r = \frac{1}{\lambda_\theta}\frac{\partial W}{\partial \lambda_r}, \quad \sigma_\theta = \frac{1}{\lambda_r}\frac{\partial W}{\partial \lambda_\theta}. \tag{5}$$

Eq. (4) is to be solved subject to the displacement boundary condition

$$r(1) = \lambda, \tag{6}$$

and the traction-free boundary condition

$$\sigma_r(\delta) = 0. \tag{7}$$

If the above boundary value problem in the limit $\delta \to 0$ has a non-trivial solution with $r(0) > 0$ for some λ, cavitation is said to occur. The above cavitation problem was solved in a series of papers by Haughton [8,41,42]. His approach was to start with a 3D strain-energy function and obtain the reduced 2D strain-energy function by using the membrane assumption $\sigma_3 = 0$ to eliminate the principal stretch λ_3 in the thickness direction. In particular, he showed that for the class of strain-energy function given by

$$W = \sum_{r=1}^{n} \mu_r (\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r,$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, sufficient conditions for non-existence of cavitation are $\alpha_1 > 1$ or

$$\alpha_1 < -1 \quad \text{and} \quad 2\alpha_n > 3 - \alpha_1,$$

but the class of strain-energy function given by