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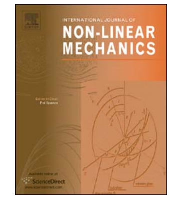

Dynamic preserving method with changeable memory length of fractional-order chaotic system

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ABSTRACT

In this paper, an asymptotically stability condition $\alpha + \beta \geq 3\gamma$ of the fractional-order Lü system is proposed by using the theory of stability. Under this asymptotically stability condition and the Riemann-Liouville fractional derivative definition, the numerical efficiency is obtained by combining the nonstandard finite difference method with the Grünwald-Letnikov method. In addition, the reported dynamic preserving properties of the nonstandard finite difference method are verified by comparing with the predictor-corrector algorithm. Moreover, in order to reduce the computation time of fractional derivatives, a model with changeable memory length of short memory principle is introduced and solved by the nonstandard finite difference method. In the numerical examples, about 30% of computation time can be reduced by applying the changeable memory length model.

1. Introduction

Recently, some numerical methods have been applied to solve the fractional-order differential equations [1–4]. The finite difference scheme in [1] is proposed to solve the fractional advection dispersion flow equations. In [2], the finite element methods have been applied to the fractional-order two point boundary value problem. An variational iteration numerical method in [3] was presented to solve some nonlinear differential equations of fractional order. In addition, the predictor-corrector (PC) algorithm in [4] is one of the most useful method among these methods. Considering these numerical methods for fractional-order differential equations, there still exist two main problems as follows: *The difficulty of computational complexity of fractional derivatives which is essentially because fractional derivatives are nonlocal operators* [5]; *Preserving the dynamic properties of the original continuous system once be discretized* [6].

How to solve these two problems* Firstly, the short memory principle (SMP) will be considered in this paper. Some methods of SMP were proposed in [5,7–9], such as the nested meshes method in [7] and the fixed memory length method in [8,9]. While the problem of the fixed memory length method is the losing of information of the chaotic system in the early part of the integral interval. Hence, the method with changeable memory length of SMP based on Grünwald-Letnikov method [10] is necessary to be considered in this paper to avoid the problem and reduce the difficulty of computation. Secondly, the nonstandard finite difference (NSFD) method proposed in [11–13] will be used in this paper, for the goal of the method is to design finite difference methods that are dynamically consistent when the continuous time models being discretized [14]. While the so-called dynamic consistency property of the NSFD method is not clear proven, but suggested by many numerical results or proved with some bounded and positive conditions at present, such as [15,14,16,17]. Therefore, the dynamic consistency property will be considered mainly in numerical in this paper, for it may beyond our knowledge to prove it analytically at present. The positive applications of the NSFD method can be found in the fields of physics, chemistry, engineering [18–20] and etc. Especially, the most attractive applications are in mathematical biology and ecology [21–24] where the merit of the NSFD method has been shown prominently. In addition, the dynamic preserving properties of the NSFD method are also well performed in solving fractional-order system [24–27]. Therefore, the NSFD method will be implemented combining with the Grünwald-Letnikov method under the Riemann Liouville definition [8], and the proposed property of the NSFD method will be verified numerically in this paper.

Without loss of generality, we employ the NSFD method to obtain numerical solutions of the fractional-order Lü system in [28] which has massive dynamic behaviors. The Lü system was introduced in [29] and considered as the bridge of the Lorenz system and Chen system. The applications of the system can be found in chaos synchronization [28,30–32]. Based on the theorem of stability [33], the asymptotic

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stability condition of the fractional-order Lü system will be obtained in this paper. In addition, the properties of the NSFD method will be investigated by comparing with the PC algorithm under the asymptotic stability condition. Moreover, in order to verify the reliability and efficiency of the proposed changeable memory length method, the corresponding numerical results are obtained via the NSFD method.

This paper is organized as follows. In Section 2, the preliminaries of the NSFD method and Grünwald-Letnikov method are presented. In Section 3, the method of changeable memory length L are derived. The construction of the NSFD scheme and analysis of stability of the fractional-order Lü System are shown in Section 4. In Section 5, the numerical results of the fractional-order Lü System are presented to verify the performance of the NSFD method and the changeable memory length method. At the end of this paper, the conclusions and acknowledgements are offered.

2. Preliminaries

2.1. The nonstandard finite difference method

The general approach of the NSFD method was first proposed by Mickens [11–13] is of simplicity in constructing the discrete scheme for the PDEs or ODEs. The NSFD discrete models are capable to replicate the properties of the exact solution of the original PDEs or OEDs with the following rules [13]:

1. The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives of the differential equations.
2. Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used.
3. Nonlinear terms must be approximated in a nonlocal way.
4. Special conditions that hold for the solutions of the differential equations should also be special discrete for the finite difference scheme.
5. The scheme should not have solutions that do not correspond to solutions of the differential equations.

Consider a given differential equation as follows:

$$\frac{dy(t)}{dt} = f(y(t)), \tag{1}$$

where $f(y(t))$ is an arbitrary continuous function. Applying the NSFD method to Eq. (1), the discrete form can be written as

$$\frac{y_{k+1} - y_k}{\phi(h)} = f(y_k) \quad (k = 1, 2, \ldots, n). \tag{2}$$

Since the form of the discrete derivative may be not unique, another form was used in [34]:

$$\frac{y_{k+1} - y_k}{\phi(h)} = f(y_k)\omega_k \quad (k = 1, 2, \ldots, n), \tag{3}$$

where

$$\omega_k = \begin{cases} 1, & \text{if } f(y_k) \geq 0, \\ \dfrac{y_{k+1}}{y_k}, & \text{if } f(y_k) < 0. \end{cases}$$

The function $\phi(h)$ in Eqs. (2)–(3) is the denominator functions of the time step size h and the $\phi(h)$ must satisfy the Eq. (4) which is in agreement with the Rule 2:

$$\phi(h) = h + O(h^2), \tag{4}$$

where $h = t/n,\ t_k = kh\ (k = 0, 1, \ldots, n)$. Here, function $\phi(h)$ can be h, $\sin(h)$, or $e^h - 1$, etc. To satisfy the Rule 3, the nonlinear term in the right hand side of Eq. (1) must be replaced by nonlocal discrete form, such as

$$y_k^2 \approx y_k y_{k+1}, \quad y_k^3 \approx \left(\frac{y_{k+1} + y_{k-1}}{2}\right) y_k^2. \tag{5}$$

Another nonlocal discrete approximations for these nonlinear terms were given by Mickens in [35] as follows:

$$a y_k^m \approx \begin{cases} (1 + a) y_k^m - y_k^{m-1} y_{k+1}, & \text{if } a > 0, \\ -(|a| + 1) y_k^{m-1} y_{k+1} + y_k^m, & \text{if } a < 0, \end{cases} \tag{6}$$

where a is the coefficients in $f(y)$. In this paper, Eq. (6) is used to approximate the nonlinear terms of the fractional-order Lü system.

2.2. The Grünwald-Letnikov method

If the following q order Riemann-Liouville fractional derivative [8] is implemented:

$${}_aD_t^q y(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t - \tau)^{m-q} y(\tau) d\tau \quad (m \leq q < m + 1,\ m \in Z), \tag{7}$$

the corresponding fractional differential equation has the form

$${}_aD_t^q y(t) = f(y(t)). \tag{8}$$

The Grünwald-Letnikov method in [10] is used to approximate the fractional derivative based on the finite differences on an uniform grid in $[0, t]$ with step h. For the sake of simplicity, assume that the function ${}_aD_t^q y(t)$ satisfies some smoothness conditions in interval $[0, t]$. Set up the grid $0 = \tau_0 < \tau_1 < \cdots < \tau_{j+1} = t = (j + 1)h$ where $h = \tau_{j+1} - \tau_j$, one can apply the notation of the finite differences

$$\frac{1}{h^q} \Delta_h^q y(t) = \frac{1}{h^q}\left[y(t_{n+1}) - \sum_{j=1}^{n} c_j^q y(t_{n+1-j}) \right], \tag{9}$$

where

$$c_1^q = q, \quad c_j^q = (-1)^{j-1}\binom{q}{j} \quad (j = 2, \ldots, n).$$

Then Grünwald-Letnikov method can be referred to [10], that is,

$${}_aD_t^q y(t) = \lim_{h\to 0} \frac{1}{h^q} \Delta_h^q y(t). \tag{10}$$

Inspired by the NSFD method, Eq. (10) can be rewritten as:

$${}_aD_t^q y(t) = \lim_{\phi(h)\to 0} \frac{1}{\phi^q(h)} \Delta_h^q y(t). \tag{11}$$

Therefore, the final NSFD approximation of fractional derivative ${}_aD_t^q y(t)$ is shown in Eq. (11).

3. The changeable memory length method

Considering the SMP, the changeable memory length method based on the Grünwald-Letnikov method is given here.

The SMP was described by *Podlubny* [8] which is commonly referred to as the *fixed memory principle* with length $L > 0$. The truncation error $\varepsilon(t)$ referred to [8,9] has the following form:

$$\varepsilon(t) \leq \frac{ML^{-q}}{|\Gamma(1 - q)|}, \tag{12}$$

where $M = \sup_{\tau\in[0,t]}|f(y(\tau))|$, L is the fixed memory length and $\Gamma(\cdot)$ denotes the Gamma function. In fact, L should be a time dependent variable, and then the upper bound of $\varepsilon(t)$ in inequality (12) is not sharp. Hence, a proposition is proposed to determine adaptive L and the numerical errors are analyzed in Section 5.3.

Proposition 1. *Let the global memory intensity* $L^f = \max_i(L_i^f)$, $L_i^f = \sum_{j=2}^{n} c_j^{q_i}$, $\forall\ q_i \in (0, 1)$, *and* $c_j^{q_i}$ *are the Grünwald-Letnikov*