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Bifurcation analysis of non-linear oscillators interacting via soft impacts

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ABSTRACT

In this paper we present a bifurcation analysis of two periodically forced Duffing oscillators coupled via soft impact. The controlling parameters are the distance between the oscillators and the difference in the phase of the harmonic excitation. In our previous paper http://arXiv:1602.04214 (P. Brzeski et al. Controlling multistability in coupled systems with soft impacts [11]) we show that in the multistable system we are able to change the number of stable attractors and reduce the number of co-existing solutions via transient impacts. Now we perform a detailed path-following analysis to show the sequence of bifurcations which cause the destabilization of solutions when we decrease the distance between the oscillating systems. Our analysis shows that all solutions lose stability via grazing-induced bifurcations (period doubling, fold or torus bifurcations). The obtained results provide a deeper understanding of the mechanism of reduction of the multistability and confirmed that by adjusting the coupling parameters we are able to control the system dynamics.

1. Introduction

Systems interacting via impacts have attracted in recent years the attention of a growing number of researchers. In many mechanical systems, such as tooling machines, walking and hopping machines or gears, the motion of some elements is limited by a barrier or the other parts of a machine. In this paper we focus on mechanical interactions produced via soft impacts [1]. Therefore we assume a finite, nonzero contact time and a penetration of the colliding bodies. The contact forces are modeled using a linear [2,3], Hertzian [4,5] or other non-linear [6] spring and a viscous damper. To describe the behavior of such systems we introduce separate sets of smooth ODEs governing the system motion during the in-contact and out-of-contact stages.

Numerous investigations have been devoted to the analysis of various dynamical phenomena induced by impacts. The characteristic bifurcation for such systems is the grazing bifurcation, which can occur both for non-impacting and impacting solutions [7–10]. The grazing bifurcation occurs when the velocity of impact is zero and the trajectory just touches the boundary of impact. Hence, when passing the grazing point the change of a control parameter causes an appearance of a new impact, which takes place with zero impact velocity (a grazing impact). Grazing bifurcations may induce different events, such as sudden loss of stability, emergence of a new orbit or multiple orbits, a change in the period of the system's motion or creation of a chaotic attractor.

In this paper we carry out a bifurcation analysis of two non-linear oscillators interacting via transient impacts. We consider system of two identical oscillators and assume the interaction starts when the distance between them is sufficiently small. When the systems are uncoupled we observe multiple stable attractors for each subsystem, so the overall system is also multistable. Therefore, in this system we are able to change the number of stable attractors and reduce the multistability via transient impacts. This phenomenon has been introduced in our previous paper [11]. In this paper we investigate the mechanism that lies behind this phenomenon and show the sequence of bifurcations which cause the destabilization of solutions.

The paper is organized as follows. In Section 2 we introduce the model of two Duffing oscillators coupled via soft impacts. The description of continuation procedure is presented in Section 3. Then, in Section 4 we show the bifurcation analysis in one and two control parameters. Finally, in Section 5 the conclusions are given.

2. Physical model of the coupled Duffing oscillators and equations of motion

We investigate two coupled Duffing oscillators schematically presented in Fig. 1. The motion of the system is governed by the following

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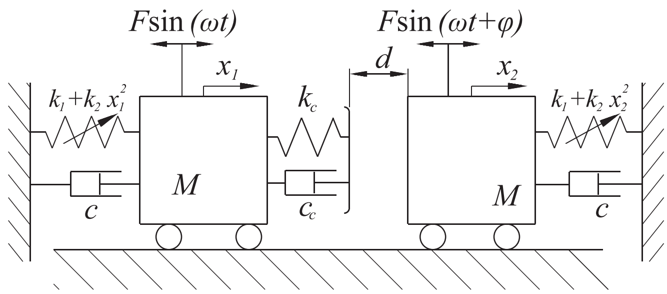

Fig. 1. Model of two discontinuously coupled Duffing oscillators.

equations:

$$M\ddot{x}_1 + k_1x_1 + k_2x_1^3 + c\dot{x}_1 + F_C = F\,\sin(\omega t), \tag{1}$$

$$M\ddot{x}_2 + k_1x_2 + k_2x_2^3 + c\dot{x}_2 - F_C = F\,\sin(\omega t + \varphi), \tag{2}$$

where a single over dot means differentiation with respect to the dimensional time. Here, F_C stands for the contact force generated by the discontinuous dissipative coupling, given by

$$F_C = \begin{cases} 0, & x_1 - x_2 < d, \\ k_c(x_1 - x_2 - d) + c_c(\dot{x}_1 - \dot{x}_2), & x_1 - x_2 \ge d. \end{cases} \tag{3}$$

In the present study, we will consider the equations of motion (1)–(2) in dimensionless form, according to the following formulas: $\tilde{x}_1 = \frac{x_1}{\ell_r}$, $\tilde{x}_2 = \frac{x_2}{\ell_r}$, $\tilde{\omega} = \frac{\omega}{\omega_r}$, $\tilde{t} = \omega_r t$, $\widetilde{M} = \frac{M}{m_r}$, $\tilde{k}_1 = \frac{k_1}{m_r\omega_r^2}$, $\tilde{k}_2 = \frac{k_2\ell_r^2}{m_r\omega_r^2}$, $\tilde{k}_c = \frac{k_c}{m_r\omega_r^2}$, $\tilde{c} = \frac{c}{m_r\omega_r}$, $\tilde{c}_c = \frac{c_c}{m_r\omega_r}$, $\tilde{d} = \frac{d}{\ell_r}$, $\widetilde{F} = \frac{F}{m_r\ell_r\omega_r^2}$.
where $\ell_r = 1[m]$, $m_r = 1\,[kg]$ and $\omega_r = 1[rad/s]$ are the reference length, mass and frequency respectively. In the rest of the paper, the results will be presented considering the nondimensional variables and parameters introduced above. Nevertheless, all tildes will be omitted for the sake of simplicity. Similarly to our previous investigation [11] the controlling parameters will be the distance d between the subsystems and the phase shift in the excitation force of the second system φ.

3. The coupled Duffing oscillators as a piecewise-smooth dynamical system

The governing equations (1)–(2) can be studied in the framework of *piecewise-smooth dynamical systems* [12]. In this context, the state space is typically divided into disjoint subregions, each defining a particular operation mode of the system, where the system behavior is described by a smooth vector field. The boundaries of the subregions are defined by the zero-set of smooth scalar functions (known as *event functions*). Event functions are usually connected to physical instantaneous events, such as: impacts, switches, transitions from stick to slip motion, etc. When a trajectory crosses the boundary of a subregion, the vector field describing the system behavior is switched according to the governing laws of the system. A boundary crossing can be accurately detected by means of e.g. the standard MATLAB ODE solvers together with their built-in event location functionality [13,14], as implemented in [15].

To study the dynamics of the coupled Duffing oscillators, we employ path-following (continuation) method, which enables to systematically explore a model response subject to parameter variations [16], with focus on the detection of possible qualitative changes in the system dynamics (bifurcations). Computational tools specialized on path-following algorithms for piecewise-smooth dynamical systems have been developed in the past, such as SlideCont [17], TC-HAT [18] (see also [19–23] for some applications of this tool) and, more recently, COCO [24,25]. In the present work, we will apply COCO to study the non-linear behavior of the coupled Duffing oscillators. The next section will explain in detail the mathematical setup required to use the continuation software in order to carry out the numerical investigation.

3.1. Modeling of the coupled Duffing oscillators in COCO

In this paper we perform numerical investigation using path-following toolbox COCO (abbreviated form of Computational Continuation Core [24]). It is a MATLAB-based analysis and development platform for the numerical solution of continuation problems. The software provides the user with a set of toolboxes that covers, to a good extent, the functionality of available continuation packages, e.g. AUTO [26] and MATCONT [27]. A distinctive feature of COCO is, however, that it offers a general-purpose framework for the user to develop specialized toolboxes that can be constructed based on a number of generic COCO-routines, common across a large range of continuation problems.

In our investigation we will use the COCO-toolbox 'hspo', which extends and improves the functionalities of the software package TC-HAT [18], an AUTO-based application for continuation and bifurcation detection of periodic orbits of piecewise-smooth dynamical systems. The main differences between these two continuation toolboxes are discussed in detail in [25]. The mathematical setup required to apply the COCO-toolbox 'hspo' is the same as for TC-HAT. It requires to divide a piecewise-smooth periodic trajectory into smooth *segments*. Each segment is then characterized by a smooth vector field describing the system behavior in the segment and an event function that defines the terminal point of the segment, as explained at the beginning of Section 3. What follows,
$\lambda{:=}(d, \varphi, \omega, F, M, k_1, k_2, c, k_c, c_c) \in \mathbb{R}_0^+ \times [0, 2\pi) \times (\mathbb{R}^+)^8$ and $u{:=}(x_1, x_2, v_1, v_2)^T \in \mathbb{R}^4$ denotes the dimensionless parameters and state variables of system (1)–(2), respectively, where $\mathbb{R}_0^+$ stands for the set of nonnegative numbers. Below, we introduce the segments that are used for the numerical implementation in COCO.

No Contact (NC). This segment occurs when the oscillating masses move without touching each other, i.e. $(x_1 - x_2 < d$. In this segment the contact force F_C equals zero (see Fig. 1). The motion of the coupled Duffing oscillators during this regime is governed by the system of equations (cf. Eqs. (1)–(2))

$$u' = f_{\mathrm{NC}}(t, u, \lambda){:=}\begin{pmatrix} v_1 \\ v_2 \\ \frac{1}{M}(F\,\sin(\omega t) - k_1x_1 - k_2x_1^3 - cv_1) \\ \frac{1}{M}(F\,\sin(\omega t + \varphi) - k_1x_2 - k_2x_2^3 - cv_2) \end{pmatrix}, \tag{4}$$

where the prime symbol denotes differentiation with respect to the nondimensional time. This segment terminates when a transversal crossing with the impact boundary defined by

$$h_{\mathrm{IMP}}(u, \lambda){:=}x_1 - x_2 - d = 0$$

is detected, and the system switches to the *Contact* segment introduced below.

Contact (C). In this operation mode the oscillating masses are in contact, i.e. $x_1 - x_2 \ge d$, which gives rise to an additional force due to the discontinuous coupling defined by the spring-damper pair (k_c, c_c). The dynamics of the system in this operation mode is described by the equations (cf. Eqs. (1)–(2))

$$u' = f_{\mathrm{C}}(t, u, \lambda){:=}\begin{pmatrix} v_1 \\ v_2 \\ \frac{1}{M}(F\,\sin(\omega t) - k_1x_1 - k_2x_1^3 - cv_1+ \\ -\,(k_c(x_1 - x_2 - d) + c_c(v_1 - v_2))) \\ \frac{1}{M}(F\,\sin(\omega t + \varphi) - k_1x_2 - k_2x_2^3 - cv_2+ \\ +\,(k_c(x_1 - x_2 - d) + c_c(v_1 - v_2))) \end{pmatrix}, \tag{5}$$

with the terminal point being defined by the event $h_{\mathrm{IMP}}(u, \lambda) = 0$, after