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Implicit equations for thermoelastic bodies

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ABSTRACT

In this paper we generalize the recent implicit models that have been put into place to describe the elastic response of bodies when thermal effects come into play. The implicit constitutive relations for thermoelastic response presented here provide a very natural way to overcome a serious problem associated with the celebrated model due to Fourier, namely infinite speed of the propagation of temperature. We also study some boundary value problems within the context of the implicit equations that we have developed. We carry out a linearization based on the classical assumption that the displacement gradient is small and obtain constitutive relations that allow the linearized strain to be a non-linear function of the stress and temperature.

1. Introduction

The celebrated eponymous equation governing the conduction of heat, that is given the status of a 'law', namely Fourier's law, is merely an approximation which in fact predicts erroneously that temperature propagates with infinite velocity. In view of the fact that the propagation has finite speed, there has been considerable interest in developing a more meaningful equation for the conduction of heat. A pioneering study in this direction is that by Cattaneo [1]. Later, Lord and Shulman [2] studied the thermoelastic response of solids wherein they sought to ensure finite wave speeds for the propagation of temperature. These early works have been followed by papers too numerous to detail, provide minor improvements or generalization to the response of viscoelastic bodies and bodies described by higher gradient theories. The thermoelastic response studied by Lord and Shulman [2] as well as the others such as Ezzat [3], consider the response of Cauchy elastic bodies (or the sub-class of Green elastic bodies) with thermal effects being taken into consideration. In this paper, we study the response of a new class of elastic bodies that are not necessarily Cauchy elastic bodies, being described by implicit constitutive relationship between the stress and the deformation gradient, when thermal effects are included. At the outset, we would like to make a case for why the new class of implicit constitutive relations to describe the response of elastic bodies is worth studying in detail. As discussed in details by Rajagopal [4–9], there are several reasons for employing a theory wherein one has an implicit relationship between the deformation gradient and the Cauchy stress. From a philosophical standpoint the theory is in keeping with the demands of causality as the deformation is a consequence of the applied traction and the resulting stress field. Such an approach also allows for the material moduli to depend on for instance the mean value of the stress, namely the mechanical pressure, a feature exhibited by many polymeric solids (see Rajagopal and Saccomandi [10]). Furthermore, it allows the strain to have a nonlinear relationship with regard to the stress even in what would be considered the 'small strain' regime, a response characteristic of many intermetallic alloys (see Rajagopal [9], Devendiran et al. [11]). Also, a Cauchy elastic body cannot describe an elastic body which exhibits limiting strains, while a fully implicit constitutive relation or a constitutive expression wherein the Cauchy-Green strain as a function of the stress models can describe such constrained response (see [4]). Moreover, while using the linearized version of such implicit theories one does not necessarily have to face glaring inconsistencies such as those encountered while studying the state of strain at a crack tip within the context of the linearized theory of elasticity. As Cauchy elastic bodies are a very special sub-class of the class of bodies characterized by the implicit constitutive relation between the stress and the deformation gradient, the classical results of thermoelasticity are recovered when attention is directed to the sub-class of Cauchy elastic bodies.

In addition to the issue of ensuring finite speed for the propagation of temperature, we also consider the counterpart to the celebrated Oberbeck-Boussinesq equations (see Oberbeck [12,13] and Boussinesq [14]) that has been developed to describe the response of fluids that can only undergo isochoric motions in isothermal processes, but can undergo compression or expansion in non-isothermal processes. The Oberbeck-Boussinesq approximation is one of the most useful approximations in fluid mechanics, and is employed to study problems in

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astrophysical and geophysical fluid dynamics as well as several other technological applications. It is important to bear in mind that the Oberbeck-Boussinesq approximation does not stem from retaining terms in a proper perturbation expansion but in fact includes terms of different orders in the same equation. A detailed discussion of the status of the Oberbeck-Boussinesq approximation within the context of the full Navier-Stokes-Fourier theory can be found in the paper by Rajagopal et al. [15]. The Oberbeck-Boussinesq approximation has been extended for various other constitutive equations governing the response of fluids (see [16,17] and the references cited therein). The basic approach to the problem is the assumption that the deformation gradient meets the restriction that the motion is isochoric in isothermal processes.

The counterpart of the above problem within the context of classical nonlinear thermoelasticity is however not straightforward. It is well known that the above constraint leads to physically unrealistic situations such as that of instability of wave propagation (see Chadwick and Scott [18], Scott [19,20], Leslie and Scott [21], Scott [22]). Since Cauchy elastic bodies are a sub-class of the general class of implicit elastic bodies, and also overlap with bodies wherein the Cauchy-Green strain is an explicit function of the stress when the relationship is invertible, for such models the physically unrealistic situation will persist. For models wherein the relationship between the Cauchy-Green strain and the Cauchy stress is not invertible we do not know if this problem will recur. This is the object of an ongoing investigation. Here, we look at the problem wherein the constitutive relation is a non-linear relationship between the linearized strain and the Cauchy stress in which case we do not have the possibility of inverting the nonlinear expression for the linearized strain in terms of the stress. It is possible that even this class of models might exhibit the physically unacceptable behaviour observed by Scott and co-workers in the case of Cauchy elastic bodies. This is also being looked into in the ongoing investigation mentioned above.

The organization of the paper is as follows. In the next section we provide a brief introduction of the kinematics and the basic equations (see Section 2) and this is followed in Section 3 where the implicit constitutive relation between the Cauchy stress, the Cauchy-Green tensor, heat-flux vector and temperature is proposed to describe the response of a thermoelastic body. The first, and a specific form of the second, laws of thermodynamics are introduced, and a generalization of the Fourier model for heat transfer by conduction is proposed. In Section 4 the special case of isotropic relations is considered, and some subclasses of constitutive relations and equations are derived from that, assuming for some cases that some of the variables are small enough; of particular interest is the case of assuming small gradient of the displacement field. In Section 5 several simple boundary value problems are analyzed, for the special case of the constitutive equation obtained assuming that the gradient of the displacement field is small. In Section 6 the constraints of incompressibility and inextensibility are studied for two of the subclasses of constitutive equations proposed in Section 4. Finally, in Section 7 concluding remarks are made.

2. Basic equations

A point in a body $\mathcal{B}$ is denoted X and in the reference configuration the point occupies the position $\mathbf{X} = \boldsymbol{\kappa}_r(X)$. The reference configuration is denoted $\kappa_r(\mathcal{B})$. In the current configuration the position of the point is denoted $\mathbf{x}$, and it is assumed that there exists a one-to-one mapping $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The current configuration is denoted $\kappa_t(\mathcal{B})$.

The displacement field, the deformation gradient, the right Cauchy-Green stretch tensor, the Lagrange strain and the linearized strain tensors are defined, respectively, as:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \nabla_r \mathbf{x}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}),$$
$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{1}$$

where we assume $J = \det\mathbf{F} > 0$, ∇_r and ∇ are the gradient operators in the reference and the current configuration, respectively.

The Cauchy stress tensor is denoted $\mathbf{T}$ and it satisfies the equation of motion

$$\rho\ddot{\mathbf{x}} = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \tag{2}$$

where ρ is the density of the body and $\mathbf{b}$ represents the body forces in the current configuration, and where we use the notation $(\dot{\ })$ for the time derivative.

The second Piola-Kirchhoff stress tensor $\mathbf{S}$ is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}. \tag{3}$$

More details about kinematics and the above definitions can be found, for example, in [23].

3. Implicit relations for thermoelastic bodies

We will be interested in studying some subclasses of the general implicit relation for a thermoelastic body (see [4,5] for the original formulation for elastic bodies)

$$\mathfrak{F}(\mathbf{S}, \mathbf{E}, \theta) = \mathbf{0}, \tag{4}$$

where θ is the absolute temperature and $\mathfrak{F}$ is a second order tensor relation. Relation (4) would be a generalization of the classical explicit model $\mathbf{S} = \mathfrak{K}(\mathbf{E}, \theta)$, where now in (4) $\mathbf{S}$ cannot be obtained in general explicitly in terms of $\mathbf{E}$. Additionally, we have added θ as one of the fundamental variables for the heat transfer problem.

The first law of thermodynamics in the reference configuration is (see, for example, [24])

$$\rho_r\dot{\epsilon} = w + \mathrm{Div}\mathbf{h}_r + \rho_r r, \tag{5}$$

where $\mathbf{h}_r$ is the heat flux in the reference configuration, ϵ is the internal energy, ρ_r is the density in the reference configuration, $w = \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}})$ is the rate of work and r the rate of heat generated internally by the body.

The dissipation d is defined as (see, for example, [25])

$$d = \theta\dot{\eta} - \dot{\epsilon} + \frac{w}{\rho_r}, \tag{6}$$

where this dissipation must satisfy the inequality

$$d \geq 0. \tag{7}$$

The heat flux must satisfy the inequality

$$-\mathbf{h}_r\cdot\boldsymbol{\gamma} \geq 0, \quad \text{where} \quad \boldsymbol{\gamma} = \theta\nabla_r\left(\frac{1}{\theta}\right). \tag{8}$$

Adding these two inequalities (7), (8) we obtain

$$\rho_r\left(\theta\dot{\eta} - \dot{\epsilon} + \frac{1}{\rho_r}w\right) - \mathbf{h}_r\cdot\boldsymbol{\gamma} \geq 0. \tag{9}$$

Let us introduce the Helmotz potential ψ, which we assume is of the form

$$\psi = \psi(\mathbf{S}, \mathbf{E}, \theta). \tag{10}$$

The relation between the Helmholtz potential and the internal energy is

$$\psi = \epsilon - \theta\eta. \tag{11}$$

From (11) we have $\dot{\epsilon} = \dot{\psi} + \dot{\theta}\eta + \theta\dot{\eta}$ and replacing in (9) we obtain

$$-\rho_r(\dot{\psi} + \dot{\theta}\eta) + w - \mathbf{h}_r\cdot\boldsymbol{\gamma} \geq 0. \tag{12}$$

For $\dot{\psi}$ we have (in index notation and Cartesian co-ordinates)