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Chiellini integrability and quadratically damped oscillators

Ankan Pandey[a], A. Ghose-Choudhury[b,*], Partha Guha[a]

[a] S N Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700106, India
[b] Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta 700009, India

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A B S T R A C T

In this paper a new approach to study an equation of the Liénard type with a strong quadratic damping is proposed based on Jacobi's last multiplier and Chiellini's integrability condition. We obtain a closed form solution of the transcedental characteristic equation of the Liénard type equation using the Lambert *W*-function.

1. Introduction

In recent times a number of articles have appeared in the literature which deal with the phenomenon of a linear oscillator subject to a quadratic damping force [1–5]. Most elementary textbooks deal with viscous damping for the obvious reason that it involves a linear dependance on the velocity of the oscillator and presents the simplest situation where an exact analytical treatment is possible. In general this involves analysis of a second-order ordinary differential equation (ODE) of the Liénard type [10], namely $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where it is assumed that f is a constant and the function $g = x$. As damping does not arise from a single physical phenomena and is itself of various kinds, e.g., material damping, structural damping, interfacial damping, aerodynamic and hydrological drag etc., therefore a different mathematical description is needed in each case. Systems like the simple harmonic oscillator and the viscously damped harmonic oscillator, both of which can be solved by elementary techniques, however, represent idealizations of real life phenomena because they ignore nonlinear aspects of the forcing term as well as the damping force.

For more realistic models applicable to problems involving hydrological drag and aerodynamics, which usually involve higher velocities, the damping force is found to be proportional to the square of the velocity. The same is also true when an immersed object moves through a fluid at relatively high Reynolds numbers [6]- the corresponding drag force is found to be proportional to the square of the velocity $v = sgn(\dot{x})\dot{x}^2$. In recent times oscillators with a non-negative real-power restoring force $F(x) = ksgn(x)|x|^{\alpha}$ and quadratic damping have also been studied by Kovacic and Rakaric [5].

The principal feature associated with quadratic damping is a discontinuous jump of the damping force in the equation of motion whenever the velocity vanishes such that the frictional force always opposes the motion. For oscillatory systems this occurs every half cycles and means that instead of a single equation of motion the latter splits into two parts depending on the sign of the velocity. Each equation has to solved separately and matched at the points where the velocity changes sign. In general solving such a system in presence of nonlinearity is a rather daunting task and only in rare cases is an exact solution to be expected. Numerical techniques on the other hand provide valuable information about the evolution of the system and its general nature.

From the mathematical point of view the construction of first integrals for systems involving a quadratic dependance on the velocity often provides interesting insights. Indeed constants of motion are the bed rock of many of the conservation principles at the heart of theoretical physics: the work-energy theorem applied to a conservative system, is perhaps the most striking and oft quoted example, as it has evolved into the principle of conservation of energy.

In this paper we examine the equation, $\ddot{x} + sgn(\dot{x})f(x)\dot{x}^2 + g(x) = 0$, in the light of several recent articles which have also dealt with the same equation [1–3]. This is a discontinuous generalization of an equation of the Liénard type involving a quadratic dependence on the velocity. In particular we show that by imposing the Chiellini condition of integrability on the functions f and g one can subsume many of the previous examples into a compact scheme. Incidentally the Chiellini condition is typically encountered in the context of integrability of the standard Liénard equation in course of its transformation to the first-order Abel equation of the first kind and also while finding a Lagrangian/Hamiltonian description of the Liénard equation [7–10].

* Corresponding author.
E-mail addresses: ankan.pandey@bose.res.in (A. Pandey), aghosechoudhury@gmail.com (A. Ghose-Choudhury), partha@bose.res.in (P. Guha).

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However, its application in the case of quadratic damping appears to be new. We show how one can derive in a systematic manner the maximum amplitudes analytically in terms of the Lambert W function and also construct a formal solution up to quadrature.

The organization of the paper is as follows. In Section 2 we review the second-order ODE with a quadratic dependence on velocity in the context of its Lagrangian/Hamiltonian description. It is shown that such a system may be interpreted as one displaying a position dependent mass function. The trajectory is explicitly displayed by numerical investigations. In Section 3 we split the ODE into two parts as mentioned above depending on the sign of the velocity $\dot{x}$ and investigate the trajectories, maximum amplitudes as well as period of oscillations. In particular we show that the periods of the cycles and the corresponding maximum amplitudes are both determined exclusively by a potential function which involves the position dependent mass function. Furthermore by invoking the Chiellini integrability condition it is possible to write down analytic formulae for the maximum amplitudes in terms of the Lambert W function [11,12], and also deduce the solution up to a quadrature.

The Lambert W function is defined as the inverse function of the mapping $x \mapsto xe^x$ and thus solves the equation $ye^y = x$. The solution is given in the form of the Lambert W function, $y = W(x)$, i.e. W satisfies $W(x)e^{W(x)} = x$. The equation always has an infinite number of solutions, most of which are complex, and W is multivalued. The examples presented here include those obtained earlier by Cveticanin [1,2].

2. The Hamiltonian in presence of quadratic velocity

Consider a second-order ODE with a quadratic dependance on the velocity given by

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0. \tag{2.1}$$

we assume $f(x)$ and $g(x)$ are such that $f(0) = g(0) = 0$ and $f(x)$ is integrable while $g'(0) > 0$. The functional form of $g(x) = g'(0)x + g_n(x)$ where $g_n(x)$ is analytic.

The Jacobi Last Multiplier (JLM) originally arose in the context of Jacobi's efforts to derive an additional first integral for a system of n first-order ODEs given $(n-2)$ conserved quantities [13,14]. It also appears in the Lie theory of infinitesimal transformations [15,16]. In addition the JLM plays a pivotal role in the context of the inverse problem of Lagrangian dynamics as it allows for the determination of the Lagrangian of a second-order ODE of the form, $\ddot{x} = \mathcal{F}(x, \dot{x})$, an aspect that has been extensively probed in [17–22]. In this context the JLM may be defined as a solution of the equation,

$$\frac{d}{dt}\log M + \frac{\partial \mathcal{F}(x, \dot{x})}{\partial \dot{x}} = 0. \tag{2.2}$$

Therefore in case of (2.1) it follows that

$$M = \exp(2F(x)), \quad \text{where} \quad F(x) = \int_0^x f(s)ds. \tag{2.3}$$

The relationship between the JLM, M, and the Lagrangian is provided by, $M = \partial^2 L/\partial \dot{x}^2$, as a consequence of which the Lagrangian of (2.1) may be expressed as

$$L = \frac{1}{2}e^{2F(x)}\dot{x}^2 - V(x). \tag{2.4}$$

The potential $V(x)$ is determined by substituting (2.4) into the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(\frac{\partial L}{\partial x}\right),$$

and comparing the resulting equation with (2.1) which immediately shows that

$$V(x) = \int_0^x e^{2F(s)}g(s)ds. \tag{2.5}$$

Using a Legendre transformation we then obtain the Hamiltonian

$$H = \frac{1}{2}e^{2F(x)}\dot{x}^2 + \int_0^x e^{2F(s)}g(s)ds. \tag{2.6}$$

It is easily verified that H is a constant of motion and the expression for the conjugate momentum, $p = e^{2F(x)}\dot{x}$, suggests that, $M = e^{2F(x)}$, serves as a position dependent mass term. In fact equations with a quadratic velocity dependance of the type considered here naturally arise in the Newtonian formulation of the equation of motion of a particle with a variable mass. Clearly then, the trajectories for arbitrary initial conditions (x_0, y_0), where $y = \dot{x}$, are given by

$$\frac{1}{2}e^{2F(x)}y^2 + V(x) = \frac{1}{2}e^{2F(x)}y_0^2 + V(x_0). \tag{2.7}$$

In terms of the canonical momentum, $p = e^{2F(x)}\dot{x}$, the Hamiltonian H becomes

$$H = \frac{p^2}{2e^{2F(x)}} + V(x). \tag{2.8}$$

Defining a new set of canonical variables

$$P := \frac{p}{e^{F(x)}} \quad \text{and} \quad Q = \int_0^x e^{F(s)}ds = \Psi(x), \tag{2.9}$$

the Hamiltonian has the appearance

$$H = \frac{1}{2}P^2 + V(\Psi^{-1}(Q)) = \frac{1}{2}P^2 + U(Q), \quad \text{where} \quad U = V \bigcirc \Psi^{-1}, \tag{2.10}$$

and corresponds to that of a particle of unit mass provided $\Psi(x)$ is invertible.

Let us consider a simple example in which $f(x) = \text{constant}$ and $g(x) = x$. In particular suppose $f(x) = 1/2$, so that $F(x) = x/2$. Then the canonical momentum and coordinate are $P = e^{x/2}y$ and $Q = 2e^{x/2}$ respectively, and $V(x) = e^x(x-1) + 1$. Thus in terms of the new coordinates the Hamiltonian has the following form

$$H = \frac{1}{2}P^2 + \frac{Q^2}{4}\ln\left(\frac{Q^2}{4} - 1\right). \tag{2.11}$$

Fig. 1 shows some of the trajectories for the Hamiltonian in equation (2.7) with different initial conditions and it is clear from these figures that the origin (0, 0) is a center.

Note that for suitable choices of the functions f and g (2.1) often exhibits the property of isochronicity and this feature has been extensively studied in [23–25].

3. Quadratic damping

It is plain that (2.1) cannot describe a system with a quadratic damping as the term involving $\dot{x}^2$ does not change sign and oppose the motion when the velocity reverses its sign. To remedy this feature it is necessary to split (2.1) into two parts and write

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \quad \dot{x} > 0, \tag{3.1}$$

$$\ddot{x} - f(x)\dot{x}^2 + g(x) = 0, \quad \dot{x} < 0. \tag{3.2}$$

Let us denote the Hamiltonians associated with these pieces by

$$H^{\pm} = \frac{1}{2}e^{\pm}2F(x)y^2 + V^{\pm}(x) \tag{3.3}$$

with the superscript $\pm$ standing for $\dot{x} = y > (<)0$. Furthermore it will be assumed that the initial point (x_0, y_0) with $(y_0 > 0)$ is such that $V^+(x_0) = 0$ and $F(x_0) = 0$. Thus when motion commences from the initial point then the trajectory is defined by $H^+ = K_0^+ = y_0^2/2$ or in explicit form

$$\frac{1}{2}e^{+2F(x)}y^2 + V^+(x) = \frac{1}{2}y_0^2.$$

This trajectory first crosses the x-axis at say $x = x_1$ when the velocity