International Journal of Non-Linear Mechanics 92 (2017) 187–196

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Magnetosonic wave propagation in a weakly diffusive plasma

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ARTICLE INFO

Keywords:
Hartmann equilibrium
Fast magnetosonic waves
Nonlinear geometric optics

ABSTRACT

The evolution of fast magnetosonic waves in a Hartmann type magnetohydrodynamic equilibrium is studied with geometric optics methods. Rays are shown either to oscillate a round a fixed line parallel to the wall, go to infinity or hit the wall at a finite time. Perturbations of the equilibrium whose frequency has the same order as the inverse of the kinetic and magnetic diffusivities of the plasma are shown to satisfy a transport equation along the rays which may be converted into a viscous Burgers one. This allows us to describe the later evolution of these perturbations, which for periodic rays decay exponentially.

1. Introduction

The description of the different types of magnetohydrodynamic waves was one of the first successes of the MHD model of the behavior of a neutral plasma. It has now become one of the main staples of basic MHD texts: see e.g. [1,2] for complete descriptions. No matter how easy is to identify the frequencies of the Alfvén or magnetosonic waves, its ulterior evolution is a much harder problem, while undoubtedly relevant both in Fusion Theory and Astrophysics. This is a common problem to many kinds of wave propagation, and often it simplifies enormously when the wavelength is much shorter than the remaining length scales of the problem. Then waves behave in a rather similar way to particles, propagating with scarce interference along rays and forming wavefronts as in classical geometric optics. In fact the theory developed to study this subject, the so-called weakly nonlinear geometric optics method, owes much to the classical description. Initiated in the nineteen sixties [3–5], the theory was much developed later. Two excellent early summaries are [6,7], and further applications may be found e.g. in [8,9]. Caustics, resonance, shock formation are some of the many problems one may find (see e.g. [10,11]) for some examples), but in the main the theory, at least for single modes and applied to equations in conservation form may be considered fairly complete. However, this is essentially a mathematical description of hyperbolic systems and it deals awkwardly with diffusion. When the diffusion coefficient has the same order as the wave period, an asymptotic analysis is still available [7], which converts the equation satisfied by the perturbation along the rays from an inviscid to a viscous Burgers equation. This will provide useful information on how the original perturbation evolves along oscillating rays, but for those that hit the wall the initial asymptotic expansion is not sufficient to deal with the boundary layer. For this we need a double expansion in terms of the main frequency and its square root [12,13]. Since the transport equation is already complex enough, we have restricted our study to oscillating rays, which is not a too severe restriction, since as we will see for a ray to strike the wall it must start form a very large height.

The scheme of this paper is as follows: first we set an equilibrium state appropriate to the Hartmann problem, i.e. a velocity vanishing at an horizontal wall and always parallel to it, and a magnetic field with a constant component transverse to the wall plus some parallel perturbation. Then we find the fast magnetosonic frequency through the eikonal equation and study the geometry of rays and wavefronts. This turns out to be highly complex, and for the purpose of making progress we assume a low beta plasma, i.e. the sound velocity is assumed much smaller than the Alfvén one. The topology of rays is then shown to be topologically simple: a single function f associated to the equilibrium quantities yields all the answers. For initial conditions lying in a fixed neighborhood of the minimum of f, rays are periodic and oscillate between the wall and a maximum height. For those to the left of this neighborhood, which correspond to rays starting from a large height, rays hit the wall in a finite time and with a positive angle. Initial conditions to the right of the neighborhood, i.e. with small initial condition, rays tend to infinity in an infinite time. For a given equilibrium, only one of the last two possibilities may hold. The results are illustrated with plots of the function f, the level curves of the first integral (which are projections of the rays), and the rays and wavefronts for certain representative values of the equilibrium parameters. The second part deals with the transport equations of the first order perturbations of the equilibrium associated to the fast magnetosonic wave. While the equations themselves are well known in general, to find explicitly the coefficients appropriate to our equilibrium and oscillating rays is the crux of the problem and no easy task. Finally we obtain a viscous Burgers equation (with time-dependent coeffi-

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http://dx.doi.org/10.1016/j.ijnonlinmec.2017.04.001
Received 23 February 2017; Received in revised form 31 March 2017; Accepted 2 April 2017
Available online 04 April 2017
0020-7462/

cients) where application of the maximum principle will show that the perturbation decays exponentially in time, with detailed estimates of the rate of decrease.

2. Frequencies, rays and wavefronts

Let us consider an equilibrium state for an MHD plasma with velocity $\mathbf{v}$, magnetic field $\mathbf{B}$, pressure P and density ρ:

$$\mathbf{v}\cdot\nabla\mathbf{v} = \nu\Delta\mathbf{v} + (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla P \tag{1}$$

$$\mathbf{0} = \eta\Delta\mathbf{B} + \nabla\times(\mathbf{v}\times\mathbf{B}) \tag{2}$$

$$0 = \nabla\cdot(\rho\mathbf{v}). \tag{3}$$

ν represents the viscosity and η the resistivity; or, in a dimensionless setting, the inverses of the kinetic and magnetic Reynolds numbers. We take the magnetic permeability as one. The domain is the half plane $y\ge 0$, and we assume (with an obvious notation) $\mathbf{v}(x,0)=\mathbf{0}$, $\mathbf{v}(x,\infty)=(v_\infty,0)$ ($v_\infty>0$ constant). We will take the equilibrium density constant (equal to 1). The specific equation of state for the pressure will not matter, since later we will assume that the sound velocity is much smaller than the Alfvén one; we only need $P=P(y)$. We look for specific equilibria such that the flow is always horizontal,

$$\mathbf{v}(x,y) = (v(y),0), \tag{4}$$

and a constant vertical magnetic field $B>0$ is imposed, with possible horizontal perturbations,

$$\mathbf{B}(x,y) = (b(y),B), \tag{5}$$

with

$$b(\infty) = \frac{db}{dy}(\infty) = 0. \tag{6}$$

Then (3) holds trivially, and (1) and (2) reduce to

$$\nu\left(\frac{d^2v}{dy^2},0\right) + \left(B\frac{db}{dy},-b\frac{db}{dy}\right) - \left(0,\frac{dP}{dy}\right) = (0,0) \tag{7}$$

$$\eta\left(\frac{d^2b}{dy^2},0\right) + \left(B\frac{dv}{dy},0\right) = (0,0). \tag{8}$$

These equations may be integrated,

$$\frac{1}{2}b^2 + P = \text{const.} \tag{9}$$

$$\nu\frac{dv}{dy} + Bb = \text{const.} \tag{10}$$

$$\eta\frac{db}{dy} + Bv = \text{const.} \tag{11}$$

(6) and (11) yield

$$\frac{db}{dy} = \eta^{-1}B(v_\infty - v), \tag{12}$$

which taken to (10), converts this into

$$\nu\eta\frac{d^2}{dy^2}(v-v_\infty) - B^2(v-v_\infty) = 0, \tag{13}$$

whose solution is the well known Hartmann profile

$$v(y) = v_\infty\left(1-\exp\left(-\frac{B}{\sqrt{\nu\eta}}y\right)\right). \tag{14}$$

Taking this to (11) and using (6),

$$b(y) = -\sqrt{\frac{\nu}{\eta}}\exp\left(-\frac{B}{\sqrt{\nu\eta}}y\right). \tag{15}$$

We may choose the new variable

$$z = \sqrt{\frac{\nu}{\eta}}\exp\left(-\frac{B}{\sqrt{\nu\eta}}y\right), \tag{16}$$

so that, calling $m=\sqrt{\nu/\eta}$, the equilibrium is given by

$$\begin{aligned} v &= v_\infty(1-m^{-1}z)\\ b &= -z. \end{aligned} \tag{17}$$

Let us review briefly some general facts about nonlinear geometric optics. We do not look for full generality, but only the notions we will need. Consider a quasilinear hyperbolic system perturbed by a small diffusivity:

$$\frac{\partial\mathbf{u}}{\partial t} + \sum_j A_j(\mathbf{x},\mathbf{u})\frac{\partial\mathbf{u}}{\partial x_j} + \mathbf{C}(\mathbf{x},\mathbf{u}) + \mathbf{E}(\mathbf{x})\Delta\mathbf{u} = \mathbf{0}, \tag{18}$$

in the vicinity of an equilibrium $\mathbf{u}=\mathbf{u}_0$. $\mathbf{E}$ represents a semidefinite negative matrix which will be assumed small in a sense to be precised later; for the present we may ignore it. For any spatial vector $\mathbf{k}$, take a fixed eigenvalue $\Lambda(\mathbf{k})$,

$$\det\left(\Lambda(\mathbf{k})I + \sum_j A_j(\mathbf{x},\mathbf{u}_0)k_j\right) = 0. \tag{19}$$

The eikonal equation associated to this eigenvalue is

$$\frac{\partial\phi}{\partial t} = \Lambda(\nabla\phi), \tag{20}$$

and ϕ is the phase of the wave with wavenumber $\mathbf{k}$ and frequency Λ. In our case the system will be the ideal MHD one, and we choose for Λ the fast magnetosonic frequency (see e.g. [1]). If $\mathbf{u}_0$ corresponds to an equilibrium state with velocity $\mathbf{v}$, pressure P, density ρ and magnetic field $\mathbf{B}$, we have

$$[\Lambda(\mathbf{k}) - \mathbf{v}\cdot\mathbf{k}]^2 = \frac{1}{2}\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right)|\mathbf{k}|^2 + \frac{1}{2}\left[\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right)^2|\mathbf{k}|^4 - 4\frac{\partial P}{\partial\rho}\frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho}|\mathbf{k}|^2\right]^{1/2}. \tag{21}$$

Rays are solutions of the system

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \nabla_{\mathbf{k}}\Lambda(\mathbf{x},\mathbf{k})\\ \frac{d\mathbf{k}}{dt} &= -\nabla_{\mathbf{x}}\Lambda(\mathbf{x},\mathbf{k}).\end{aligned} \tag{22}$$

The phase is constant along rays,

$$\frac{d}{dt}(\phi(t,\mathbf{x}(t))) = 0. \tag{23}$$

The ray equations may be set in terms of the normalized frequency and wave vector

$$\mathbf{n} = \mathbf{k}/|\mathbf{k}|,\quad c(\mathbf{n}) = \frac{\Lambda(\mathbf{k})}{|\mathbf{k}|}. \tag{24}$$

When equations (22) are specified for the plane, they may be written in terms of c, $\mathbf{n}$ and its orthogonal $\mathbf{n}^\perp$, chosen so that $\{\mathbf{n},\mathbf{n}^\perp\}$ form an orthonormal positive system:

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + (\mathbf{n}^\perp\cdot\nabla_{\mathbf{n}}c)\mathbf{n}^\perp, \tag{25}$$

$$\frac{d\mathbf{n}}{dt} = -(\mathbf{n}^\perp\cdot\nabla_{\mathbf{x}}c)\mathbf{n}^\perp. \tag{26}$$

The fast magnetosonic frequency $c(\mathbf{n})$ satisfies

$$[c(\mathbf{n}) - \mathbf{v}\cdot\mathbf{n}]^2 = \frac{1}{2}\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right) + \frac{1}{2}\left[\left(\frac{\partial P}{\partial\rho} + \frac{B^2}{\rho}\right)^2 - 4\frac{\partial P}{\partial\rho}\frac{(\mathbf{B}\cdot\mathbf{n})^2}{\rho}\right]^{1/2}. \tag{27}$$

This equation may be written in terms of the speed of sound $c_s^2 = \partial P/\partial\rho$,