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Wave patterns in a nonclassic nonlinearly-elastic bar under Riemann data☆

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ABSTRACT

Recently there has been interest in studying a new class of elastic materials, which is described by implicit constitutive relations. Under some basic assumption for elasticity constants, the system of governing equations of motion for this elastic material is strictly hyperbolic but without the convexity property. In this paper, all wave patterns for the nonclassic nonlinearly elastic materials under Riemann data are established completely by separating the phase plane into twelve disjoint regions and by using a nonnegative dissipation rate assumption and the maximally dissipative kinetics at any stress discontinuity. Depending on the initial data, a variety of wave patterns can arise, and in particular there exist composite waves composed of a rarefaction wave and a shock wave. The solutions for a physically realizable case are presented in detail, which may be used to test whether the material belongs to the class of classical elastic bodies or the one wherein the stretch is expressed as a function of the stress.

1. Introduction

Until recently, models used to describe the elastic response of bodies belonged to either the class of Cauchy elastic bodies or Green elastic bodies. Recently, Rajagopal [23,24] introduced a much larger class of elastic bodies that included Cauchy elastic bodies and Green elastic bodies as a subset, if by elastic response one refers to a response wherein the body is incapable of dissipating energy, that is, inability to convert working into thermal energy. Of particular reference to the current work are bodies defined by implicit constitutive relations between the stress and the deformation gradient, or the sub-class wherein the strain in the body is a function of the stress. Such models are relevant when one has a material wherein the body exhibits a limiting strain or when the response between the strain and stress becomes non-linear even for very small strains wherein the classical models of elasticity reduce to the classical linearized elastic model. When the elastic body exhibits limiting strain then one could encounter the possibility that the stress cannot be expressed as a function of the strain (see Rajagopal [23]). A detailed mathematical treatment of such a response can be found in Bulicek et al. [4]. With regard to the possibility of a non-linear relationship between the strain and the stress, even when the strains are very small, one needs but look at the response of alloys such as Gum metal (see Saito et al. [28]) and many other Titanium Nickel based alloys (see Talling et al. [31], Withey [35], Zhang [36]). The response of such alloys cannot be described by the classical linearized elastic response but can be described very well with the help of the new class of elastic models wherein the linearized strain is a non-linear function of the stress (see Rajagopal [26]). Another very important class of problems where the new class of models might prove to be very useful is in predicting the state of strain in the neighborhood of cracks and the tips of notches, etc. While the linearized theory of elasticity predicts strains that blow up in the neighborhood of the tip of a crack, contradicting the very precepts under which the approximation is derived, the new class predicts results that are physically meaningful in that the strains are bounded and never exceed the limit of small strain that is supposed (see Rajagopal and Walton [27], Kulvait, et al. [14]).

Nonlinear waves in elastic bars, within the traditional framework that the stress is a function of the strain, have been studied in various contexts. For example, recently Huang, Dai, Chen and Kong [11] showed that for certain nonlinearly elastic materials, it is possible to generate a phenomenon in which a tensile wave can catch the first transmitted compressive wave (so the former can be undermined) in an initially stress-free two-material bar. Depending on the interval of the

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initial impact, the wave catching-up phenomena can happen in two wave patterns. Some asymptotic solutions were also constructed. As a continuation of this work, Huang, Dai and Kong [10] investigated the wave catching-up phenomenon in a nonlinearly elastic prestressed two-material bar and the global structure stability of nonlinear waves was also proved by the method of characteristics and the theory of typical boundary problems. An interesting study on impact-induced phase transformation in a shape memory alloy rod was carried out by Chen and Lagoudas [6], and notably they also found that composite waves with a rarefaction wave and a shock wave can arise.

In this paper, we study the Riemann problem for a specific sub-class of the new class of elastic bodies proposed by Rajagopal and focus on the various wave patterns. These equations do not possess convexity though they are strictly hyperbolic. In this study, the Reimann problem for this special sub-class is solved completely. We find that, depending on the initial condition, a variety of wave patterns can arise including a composite wave comprising of a rarefaction wave and a shock wave. We also note that due to the implicit constitutive relation (5), it is natural to select the velocity and the stress as the unknowns. Within such a framework, the equations of motion governing the sub-class of bodies under consideration cannot be written in terms of the type of conservation laws that hold for the classical elastic body.

To introduce the kind of constitutive relation adopted in this paper, we first recall some basic definitions in kinematics. The reference configuration, denoted by B, is assumed to be stress-free. A particle $\mathbf{X}\in B$ occupies the position $\mathbf{x}\in B_t$, where B_t is the configuration at time t, that is referred to as the current configuration. The mapping that maps the reference configuration to the current configuration is assumed to be one to one, and is given by $\mathbf{x}=\chi(\mathbf{X}, t)$. We denote the displacement by $\mathbf{u}=\mathbf{x}-\mathbf{X}$. Then the gradients of displacement are given as

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}}=\nabla_{\mathbf{X}}\mathbf{u}=\mathbf{F}-\mathbf{I} \quad \text{or} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{x}}=\nabla_{\mathbf{x}}\mathbf{u}=\mathbf{I}-\mathbf{F}^{-1},$$

where $\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ is the deformation gradient tensor, and $\mathbf{I}$ is the identity tensor. The Green-Saint Venant strain $\mathbf{E}$ is given by

$$\mathbf{E}=\frac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u}+(\nabla_{\mathbf{X}}\mathbf{u})^T+(\nabla_{\mathbf{X}}\mathbf{u})^T\nabla_{\mathbf{X}}\mathbf{u}). \tag{1}$$

When one assumes that the displacement gradient is small so that the last term that appears in the right hand side of (1) can be ignored in comparison to the other terms, one obtains the linearized measure of strain. The constitutive relation for elastic response within the classical theory of Cauchy or Green elasticity then leads to the popular approximation of linearized elasticity. Recently, Rajagopal [23] (see also Rajagopal [24–26]) introduced the following implicit constitutive relation for isotropic elastic materials

$$\mathbf{f}(\mathbf{T}, \mathbf{B})=\mathbf{0}, \tag{2}$$

where $\mathbf{B}=\mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green strain tensor and $\mathbf{T}$ is the Cauchy stress tensor. The general class (2) includes Cauchy elastic bodies as a special sub-class and another special subclass that is useful and is given by

$$\mathbf{B}=\tilde{\alpha}_0\mathbf{I}+\tilde{\alpha}_1\mathbf{T}+\tilde{\alpha}_2\mathbf{T}^2, \tag{3}$$

where the materials moduli $\tilde{\alpha}_i$ $(i=1, 2, 3)$ depend on the density and the principal invariants of the Cauchy stress. Under the small strain assumption

$$\max_{\mathbf{X}\in B, t\in\mathbb{R}} \|\nabla_{\mathbf{X}}\mathbf{u}\|=O(\delta), \quad \delta\ll 1,$$

where $\|\cdot\|$ denotes the trace norm, Rajagopal [23] obtained the approximation with $O(\delta)$ from (3) as follows

$$\boldsymbol{\epsilon}=\alpha_0\mathbf{I}+\alpha_1\mathbf{T}+\alpha_2\mathbf{T}^2,$$

where as usual the materials moduli α_i $(i=1, 2, 3)$ depend on the density in current configuration and the principal invariants of Cauchy stress, ϵ is the linearized strain tensor. In particular, Kannan, Rajagopal and Saccomandi [12] proposed the following special constitutive relation:

$$\boldsymbol{\epsilon}=\beta(\mathrm{tr}\mathbf{T})\mathbf{I}+\alpha\left(1+\frac{\gamma}{2}\mathrm{tr}\mathbf{T}^2\right)^n\mathbf{T}, \tag{4}$$

where $\alpha\geq 0$, $\beta\leq 0$, $\gamma\geq 0$ and n are constants.

There have been many studies carried out within the context of the new class of elastic bodies defined by (4). Of relevance to the current study is the paper by Kannan, Rajagopal and Saccomandi [12], wherein they investigated the unsteady motions of this new class of elastic solids. It was shown that the stress wave changes its shape since the wave speed depends on the stress and the value of stress varies according to the thickness of the slab. All these phenomena for the generated stress wave are quite different from what one observes for a classical linear elastic material.

When we restrict the constitutive relation (4) to one dimension, we obtain the one-dimensional constitutive relation

$$\epsilon=\beta T+\alpha\left(1+\frac{\gamma}{2}T^2\right)^n T. \tag{5}$$

We will assume that the constants in (5) satisfy that

$$\alpha>0, \quad \beta<0, \quad \gamma>0, \quad n>0. \tag{6}$$

Moreover, we suppose that

$$\alpha+\beta>0. \tag{7}$$

Remark 1. The assumption (7) guarantees the following governing system of equations (8) is hyperbolic.

In this paper, we consider the Riemann problem for nonlinear wave equations

$$\rho\frac{\partial v}{\partial t}=\frac{\partial T}{\partial x}, \quad \frac{\partial \epsilon}{\partial t}=\frac{\partial v}{\partial x} \tag{8}$$

with the initial data

$$(T, v)(0, x)=\begin{cases}(T_l, v_l), & x<0,\\ (T_r, v_r), & x>0,\end{cases} \tag{9}$$

where t, x represent the time and spatial coordinate respectively, ρ the density of elastic body, T the Cauchy stress, ϵ the strain, v the particle velocity. The constant Riemann data in (9) satisfy that $(T_l, v_l)\neq(T_r, v_r)$.

Riemann problem for PDEs is of significance not only in physics, but also in mathematics. It is well-known that the Riemann problem can be used as a building block to prove existence results for the Cauchy problem for (8) with general initial data [9], possibly having large total variation [3].

For the gas dynamics equations with convex condition, the Riemann problem has been well-studied (see [5,29]). Wendroff [33,34] investigated the gas dynamics equations without convexity conditions for the pressure and constructed a solution to the Riemann problem. Liu [18,19] considered the Riemann problem for general systems of conservation laws. By introducing an extended entropy condition, which is equivalent to the Lax's shock inequalities [15] when the system is genuinely nonlinear, Liu [19] proved the uniqueness theorem for the Riemann problem of the gas dynamics equations without convexity conditions for the pressure. By a special vanishing viscosity method, Dafermos [7] obtained the structure of solutions of the Riemann problem for a general 2×2 conservation laws. Matsumura and Mei [21] considered the nonlinear asymptotic stability of viscous shock profile for a one-dimensional system of viscoelasticity, where the constitutive relation is non-convex. They applied the degenerate shock condition proposed by Nishihara [22] to single out an admissible shock solution. By introducing a generalized shock in [22], Sun and Sheng [30] constructed the solutions to the Riemann problem for a system of nonlinear degenerate wave equations in elasticity, for which the strain-