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Submodels of model of nonlinear diffusion with non-stationary absorption

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ABSTRACT

We study the model, describing a nonlinear diffusion process (or a heat propagation process) in an inhomogeneous medium with non-stationary absorption (or source). We found tree submodels of the original model of the nonlinear diffusion process (or the heat propagation process), having different symmetry properties. We found all invariant submodels. All essentially distinct invariant solutions describing these invariant submodels are found either explicitly, or their search is reduced to the solution of the nonlinear integral equations. For example, we obtained the invariant solution describing the nonlinear diffusion process (or the heat distribution process) with two fixed "black holes", and the invariant solution describing the nonlinear diffusion process (or the heat distribution process) with the fixed "black hole" and the moving "black hole". The presence of the arbitrary constants in the integral equations, that determine these solutions provides a new opportunities for analytical and numerical study of the boundary value problems for the received submodels, and, thus, for the original model of the nonlinear diffusion process (or the heat distribution process). For the received invariant submodels we are studied diffusion processes (or heat distribution process) for which at the initial moment of the time at a fixed point are specified or a concentration (a temperature) and its gradient, or a concentration (a temperature) and its rate of change. Solving of boundary value problems describing these processes are reduced to the solving of nonlinear integral equations. We are established the existence and uniqueness of solutions of these boundary value problems under some additional conditions. The obtained results can be used to study the diffusion of substances, diffusion of conduction electrons and other particles, diffusion of physical fields, propagation of heat in inhomogeneous medium.

1. Introduction

Mechanics is a science of motion and equilibrium of bodies and continuum media under the influence of various forces of nature, and of the interactions, the processes that accompany these movements. One of the main methods of modern mechanics is the creation and research of models of the phenomena studied.

The models are constructed not only for the objects of study, but also how they interact, ie it is the models of various kinds of forces and fields, the heat fluxes, diffusion, etc. The model is a representation (scheme) phenomena more simple than the original, but it reflects the basic properties of this phenomena. Mathematical model is a description of the scheme by mathematical language. Many mathematical models of continuum mechanics are formulated in the form of linear and quasi-linear differential equations. In the derivation of these equations mechanics and physics used their invariance under transformations, as a consequence of the symmetry of space-time, which describes the phenomenon. The set of transformations acts in the space-time around us that allows to represent the geometric structure of the space. For example: the homogeneity (independence of the properties of the space from the place), the isotropy (independence of direction), the dynamic similarity of the events (Galileo and Lagrange invariance). The set of transformations acts in the space-time around us that allows us to represent the geometric structure of space. For example: homogeneity (independence properties of the space of the place), isotropic (independent of direction), the dynamic similarity of events (Galileo and Lagrange invariance). But symmetry pattern can be hidden, they may be the result of physical properties of the phenomena being modeled. The use of symmetry properties allow correctly simulate the phenomena and to classify the submodels.

The symmetry analysis of the equations of models of mechanics of continuous media is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes. The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations of model for the group classification of the model, for the research of the obtained submodels, for the research of the exact solutions.

In this paper by the methods of symmetry analysis we study the model of nonlinear diffusion in an inhomogeneous medium with the non-stationary absorption (or source). This model is described by the

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equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x^{\alpha}u^{\beta}\frac{\partial u}{\partial x}\right) - \varphi(t)u, \tag{1}$$

where $u = u(t, x)$ is a substance concentration at the point $x \in (-\infty, \infty)$ at the time t; α, β are parameters of diffusion coefficient $D = x^{\alpha}u^{\beta}$, characterizing its velocity; $\varphi(t)u$ is non-stationary absorption (or source) coefficient.

We are assume further that $\varphi(t)$, α and β satisfy to the condition

$$\alpha\beta\varphi'(t) \neq 0. \tag{2}$$

This condition means that the diffusion is nonlinear; the medium is inhomogeneous; for $\varphi(t) > 0$ there is absorption, and for $\varphi(t) < 0$ there is an external source.

Note. *Eq. (1) with the condition (2), also describes the nonlinear process of the heat propagation in an inhomogeneous rod in the presence of a non-stationary absorption (for $\varphi(t) > 0$) and a non-stationary heat source (for $\varphi(t) < 0$). In this case, $u = u(t, x)$ is the temperature at the point x of the rod at the moment of time t.*

Symmetry properties and the simplest invariant solutions of the Eq. (1) for some particular values of the parameters α, β without absorption or with stationary absorption were studied in many papers (see for example, [1–12]). In particular, [12] is devoted to a discussion of the reduction methods for evolution equations based on invariant surface conditions related to functional separation of variables. The authors propose to use for this a direct, an inverse, and a mixed method. For group classification can be used, of course, only the direct method. In fact, it is nothing more than a different presentation of the classical method of finding of the group, admitted by the differential equation, with added coupling equation. Applications to the diffusion equations in [12] are devoted to the equations with stationary absorption (or source). However in many processes of diffusion and heat propagation, the absorption (or source) is non-stationary. The articles related to the group classification and study by methods of the symmetry analysis for the diffusion equation with non-stationary absorption (or source) almost absent. The reason is that the group classification of the diffusion equation even with homogeneous non-stationary absorption (or source) is much more difficult problem than the group classification of the equation with a stationary absorption (or source). The problem of the group classification of the diffusion equation with nonstationary inhomogeneous absorption (or source) is even more difficult.

Symmetry properties and invariant solutions of the equation describing nonlinear diffusion in an inhomogeneous medium with non-stationary absorption (or source) of the special kind were studied in [13]. Symmetry properties of the Eq. (1), under the condition (2), previously have not been studied. Also, invariant solutions and the boundary value problems, studied in the present paper have not been studied, and nonlinear integral equations, to which are reduced these boundary value problems have not been obtained and investigated.

The main objective of this study is to obtain all the submodels of the model (1), having different group properties, and to obtain the most complete description of the set of solutions for each of these submodels. For each found solution we will indicate its possible use for the solution of the boundary value problem having the physical meaning. A mechanical relevance of these solutions is as follows: 1) these solutions describe the nonlinear diffusion process (or the heat propagation process) in an inhomogeneous medium with non-stationary absorption (or source), 2) these solutions can be used as test solutions in numerical calculations performed in the studies of the processes of the nonlinear diffusion (or the heat propagation) in an inhomogeneous medium with non-stationary absorption (or source).

2. Group properties

The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations of model for the group classification of the model, for the research of the obtained submodels, for the research of the exact solutions. Mathematical models of many phenomena of the real world and, of course, models of continuum mechanics are formulated in the form of linear and quasi-linear differential equations. In the derivation of these equations mechanics and physics used their invariance under transformations, as a consequence of the symmetry of space-time, which describes the phenomenon. The set of transformations acts in the space-time around us that allows us to represent the geometric structure of the space. For example: the homogeneity (independence of the properties of the space from the place), the isotropy (independence of direction), the dynamic similarity of the events (Galileo and Lagrange invariance). But the symmetries of the model can be hidden, they may be a result of the physical properties of the modeled phenomenon. Using of all symmetry properties allow us correctly to simulate the phenomenon and to classify the submodels. Symmetry analysis of these equations is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes. For example, in [14] it was studied the structure of the groups of symmetries of some mathematical models of continuum mechanics in order to clarify the physical meaning of admitted symmetries.

2.1. Group classification

We will fulfill group classification of Eq. (1). We will solve the problem of the group classification of this equation using the algorithm proposed in [15,16]. This algorithm has been successfully used in [10,11,13,17–21] for group classification of the various equations of mechanics and mathematical physics.

An arbitrary element of this equation is $\boldsymbol{f} = (\alpha, \beta, \varphi)$. Structure equations of an arbitrary element are written as follows:

$$\boldsymbol{f}_x = 0, \quad \boldsymbol{f}_u = 0, \quad \alpha_t = 0, \quad \beta_t = 0. \tag{3}$$

The operator of generalized equivalence transformations of the Eq. (4) is defined as

$$\xi^0(t, x, u)\partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u + \boldsymbol{\zeta}(t, x, u, \boldsymbol{f})\cdot\partial_{\boldsymbol{f}},$$

where ξ^0, ξ^1, η, $\boldsymbol{\zeta}$ are smooth functions of their variables.

The condition of invariance of the manifold determined by the Eqs. (1), (3) to this operator, with allowance [15,16] for the rule of extension of this operator after splitting in terms of parametric derivatives yields a system of the equations determining the generalized equivalence transformations of Eq. (1) and the specializations of the arbitrary element $\boldsymbol{f}$.

Solutions of this overdetermined system are all specializations of the arbitrary element and the corresponding equivalence transformations of the Eq. (1). These equivalence transformations form the set of generalized equivalence transformations of the Eq. (1). For the Eq. (1) the set of the generalized equivalence transformations of this equation coincides with the group of its universal equivalence transformations. For the specializations of the arbitrary element we study the action of the group of equivalences of the Eq. (1) with this arbitrary element or, more exactly, the action of the factor-group of this group of equivalences by the kernel of the main groups of the Eq. (1) on the Eq. (1) with this arbitrary element. As a result of this action, equivalent equations are formed. To find all non-equivalent equations, we construct an optimal system of subgroups for the considered group of equivalences or, more exactly, for the factor-group of this group of equivalences by the kernel of the main groups of the Eq. (1). The equivalence transformations acting on $\boldsymbol{f}$ identically form the kernel of the main groups of the Eq. (1) with this arbitrary element $\boldsymbol{f}$, i.e., they are admitted by the Eq. (1) for all elements $\boldsymbol{f}$ possessing the considered arbitrariness. In addition to the kernel of the main groups, the Eq. (1) admits each subgroup of the group of equivalences under the condition