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On the physics of viscoplastic fluid flow in non-circular tubes

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ABSTRACT

Flow of Bingham plastics through straight, long tubes is studied by means of a versatile analytical method that allows extending the study to a large range of tube geometries. The equation of motion is solved for general non-circular cross-sections obtained via a continuous and one-to-one mapping called the shape factor method. In particular the velocity field and associated plug and stagnant zones in tubes with equilateral triangular and square cross-section are explored. Shear stress normal to equal velocity lines, energy dissipation distribution and rate of flow are determined. Shear-thinning and shear-thickening effects on the flow, which cannot be accounted for with the Bingham model, are investigated using the Hershey-Bulkley constitutive formulation an extension of the Bingham model. The existence and the extent of undeformed regions in the flow field in a tube with equilateral triangular cross-section are predicted in the presence of shear-thinning and shear-thickening as a specific example. The mathematical flexibility of the analytical method allows the formulation of general results related to viscoplastic fluid flow with implications related to the design and optimization of physical systems for viscoplastic material transport and processing.

1. Introduction

Knowledge of the flow of viscoplastic materials is relevant in many contexts such as flow of paints, pastes, suspensions in complex geometries with industrial applications, coating and mining, foodstuffs processing, cosmetic, and pharmaceutical and construction industries, ceramics extrusion, blood and other biological fluid flows, semi-solid materials and in some natural flows such as mud, lava displacements and debris flow. In all these applications as well as natural phenomena the rate of flow, the velocity distribution and the energy dissipation are important flow variables to determine. One model of viscoplastic fluid widely used is Bingham model for its capacity of predicting useful results in most areas of interest. The Bingham model becomes non-linear for flow configurations different from parallel axisymmetric or unbounded parallel surfaces and, moreover, requires careful interpretation and analysis of mathematical results, which are meaningful only when all physically relevant conditions between stress and rate of deformation are met. Other types of non- Newtonian bounded flows, such as viscoelastic fluid flows in tubes, may be fully described physically over the whole flow region by means of mathematical or numerical results derived from the constitutive and linear momentum balance equations. But this is not the case for viscoplastic flows described by the Bingham model since it predicts physically meaningless results in some zones that must be identified and characterized as plug zones and stagnant zones where there is no deformation. This is not explicitly predicted by the Bingham model, and must be deduced from conditions associated with the yield stress, tube contour and the related physical considerations. Understanding the dynamics of the formation of dead regions for instance is important to the design of extrusion geometries. It is quite difficult to model viscoplastic fluid flow and design operating systems in most real-life contexts. In particular the determination of the location and shape of the boundary separating the yielded and unyielded masses of the fluid must be part of the solution of the initial boundary value problem.

Several authors have addressed in the past the analysis of the flow of Bingham fluids in conduits and related geometries. The ground-breaking work of Russian researchers in the sixties set the tone for the research direction for decades to come. Safronchik [1–3] and Mosolov and Mjasnikov [4–6] conducted fundamental investigations on the propagation and the location of the yield surface and its properties and the plug and dead regions in the flow, respectively. The channel flow of a Bingham plastic with a given initial velocity distribution and a time dependent pressure gradient is investigated in [1] to determine the subsequent velocity field and the location of the yield surface. A highly complicated equation for the velocity dependent on the location of the interface between the plug zone and the flowing mass of the fluid is

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derived, and a non-linear integral equation for the position of the interface is presented. The unsteady motion of a Bingham fluid both in bounded and unbounded domains is investigated in [2], specifically the unsteady flow field generated by a rotating cylinder and the flow field in a Couette device, respectively. A non-linear integral equation to determine the location of the yield surface is derived but left unsolved. The unsteady flow of a Bingham fluid in a circular tube is examined in [3] and again an unsolved complicated equation is presented to determine the location of the yield surface. These equations can be solved albeit with considerable difficulty; however whether or not their predictions lead to the correct physical solution, the speed of propagation and position of the yield surface, is an open issue as can be seen from the discussion in the papers of Huilgol [7,8]. For instance the attempt by Huilgol [7] to solve the non-linear integral equation derived by Safronchik [1] to determine the speed and location of the yield surface in the Rayleigh problem for a Bingham fluid led to the non-existence of a solution. Huilgol states that the reasons behind this non-physical result are the importance of the homogeneous and non-homogeneous boundary conditions imposed on the velocity field at the yield surface, and points out that a qualitative understanding of the existence or non-existence of moving yield surfaces in any flow of yield stress fluids requires a deeper insight into the right conditions to be imposed at the interface. Conditions for the existence of the plug zones were investigated by Huilgol [8] who showed in particular the singularity of the yield surface across which the velocity, the acceleration and the velocity gradient as well as the shear stress, its time derivative and its gradient are all continuous, but the time derivative of the acceleration, the spatial gradient of the acceleration, the second gradient of the velocity and the corresponding temporal and spatial gradients of second order of the shear stress all undergo jumps.

The pioneering work of Safronchik [1–3] and the work of Huilgol [7,8] shows how difficult it is to determine the location and speed of propagation of a yield surface even in simple geometries. The complexity of the behavior of the yield surface was further brought to light by Glowinski [9] who showed that the yield surface may move laterally with a finite speed in the pressure gradient driven flow of a Bingham fluid, the rigid core in the center gradually becoming larger, the yield surface expanding with a finite speed of propagation, decelerating the fluid and eventually choking off the flow and bringing it to rest. Specifically he proved that the flow of a Bingham fluid in a pipe of arbitrary cross-section comes to a halt in a finite amount of time if the pressure gradient drops suddenly below a critical value needed to overcome the effect of the yield stress in contrast to the behavior of a Newtonian fluid which comes to rest in an infinite amount of time when the pressure gradient is removed suddenly.

Sekimoto [10,11] made important contributions in the early nineties to the determination of the propagation speed and the location of the yield surface. He finds in [10] a similarity solution and derives an equation for the location of the yield surface in the case of an existing steady simple shear flow in the semi-infinite region over a flat plate which becomes unsteady due to a sudden reduction in the shear stress on the boundary to a value below the yield stress of the Bingham fluid. He shows that the yield surface propagates from the flat plate boundary into the fluid and derives an evolution equation for the location of the yield surface at subsequent times; however he does not solve the equation either. The lateral movement of a yield surface in a shearing flow is considered in [11]. He correctly assumes that the lateral motion of the yield surface obeys the diffusion equation,

$$\rho\frac{\partial u}{\partial t}=\frac{\partial \tau_{xy}}{\partial y}$$

and asserts that the gradient of the shear stress τ_{xy} is continuous across the yield surface provided certain continuity conditions are met the continuity of the local acceleration among them. He presents an evolution equation for the location of the yield surface; however it is not solved and no examples are given.

The existence and uniqueness of the plug region was shown by Mosolov and Mjasnikov [4] using variational methods. In particular they show exactly and analytically that the flow stops if the Bingham number Bi exceeds a critical Bingham number $Bi_c=\frac{4}{2+\sqrt{\pi}}=1.0603178$ with no-slip conditions on the walls when the cross-section is a square. This result, that there is a limiting Bingham number for a given cross-sectional tube, was extended later to general cross-sectional tubes by Duvaut and Lions [12], and much later was confirmed through extensive numerical computations by Saramito and Roquet [13] for a square cross-section with no-slip conditions as well as with slip on the walls Roquet and Saramito [14]. In the latter case Bi_c depends on a dimensionless slip parameter S that quantifies the extent of slip. For S =0.6 the critical Bingham number is determined to be $Bi_c=\frac{2}{2+\sqrt{\pi}}=0.5301589$. When $S=+\infty$ fluid sticks to the wall and no-slip condition prevails. The existence of dead regions with a concavity always turned towards the inside of the cross-section was proven analytically by Mosolov and Mjasnikov [5]. This is clearly a very difficult proposition to show numerically as a general statement as is obvious from the numerical results presented by Saramito and Roquet [13] and Roquet and Saramito [14].

The first numerical study of yield stress fluids was done by Fortin [15]. The augmented Lagrangian method framework introduced by Fortin and Glowinski [16] and further developed by Glowinski and LeTallec [17] was used by Huilgol and Panizza [18] to study the flow in an annulus and an L-shaped cross-sectional tube. The flow of a Bingham fluid in a square duct, and in particular the geometry of the plug zones as well as the dead zones at the duct corners was explored numerically by Saramito and Roquet [13] and Roquet and Saramito [14]. In the earlier paper [13] the fully developed Poiseuille flow of a yield stress fluid in a square cross-section was studied with no-slip condition at the walls using the augmented Lagrangian method framework coupled to a mixed anisotropic auto-adaptive finite element method. In the later paper [14] the Poiseuille flow of a yield stress fluid in a square cross-section with slip yield boundary condition was considered numerically using the same approach introduced by the authors previously. The consideration of slip is important as it frequently occurs in the flow of two-phase systems such as suspensions, emulsions and in industrial viscoplastic flow problems such as concrete pumping, and it appears to be more pronounced when the material has a yield stress property such as bio-solids and pastes. Steady flow of Bingham fluids in narrow eccentric annuli was investigated both analytically and numerically by Walton and Bittleston [19], and conditions for the existence of plug zones and quasi plug zones were discussed in the context of the flow of pastes and suspensions in complex geometries with industrial applications. Wachs [20] also studied the problem for a wide range of the relevant parameters using numerical methods. The present authors [21,22] studied viscoplastic flows in a variety of non-circular tubes both analytically and numerically relating geometric and flow variables to predict velocity distribution, rate of flow, and energy dissipation. Recent work that contributes insights into the physics of the flow of viscoplastic fluids and the complexities associated with it, as constitutively characterized by the Bingham model, addresses thermal effects, Akram et al. [23] and Turan et al. [24], and the Lattice Boltzmann method applications in complex geometries, Tang et al. [25]. A major difficulty with the flow of a Bingham fluid in complex geometries is the existence of spatial discontinuities in kinematic and dynamic variables at the interface between regions undergoing deformation and the plug and stagnant (or dead) zones.

To the best knowledge of the present authors, no published analytical work exists shedding light onto the general behavior of flows of Bingham fluids in non-circular tubes of arbitrary cross-section. The kinematics and dynamics of the steady, developed and isothermal flow of Bingham fluids in longitudinally constant, non-circular cross-