International Journal of Non–Linear Mechanics 88 (2017) 67–73

Contents lists available at ScienceDirect

International Journal of Non–Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Modified von Kármán equations for elastic nanoplates with surface tension and surface elasticity

Y.M. Yue[a,b], C.Q. Ru[b], K.Y. Xu[a,*]

[a] Shanghai Institute of Applied Mathematics and Mechanics, Shanghai Key Laboratory of Mechanics in Energy Engineering, Department of Mechanics, Shanghai University, Shanghai 200072, China

[b] Department of Mechanical Engineering, University of Alberta, Edmonton, Canada T6G 3G8

ARTICLE INFO

Keywords:
Kirchhoff plate
Surface tension
Surface stress
Von Kármán equations
Postbuckling
Free vibration

ABSTRACT

In this paper, modified von Kármán equations are derived for Kirchhoff nanoplates with surface tension and surface tension-induced residual stresses. The simplified Gurtin-Murdoch model which does not contain non-strain displacement gradients in surface stress-strain relations is adopted, so that the von Kármán strain-compatibility equation can be expressed in terms of the stress function and deflection. The modified von Kármán equations derived here are different than the existing related models especially for elastic plates with in-plane movable edges. Unlike the existing models which predict a surface tension-induced tensile pre-stress for an elastic plate with in-plane movable edges, the present model predicts that this tensile pre-stress is actually cancelled by the surface tension-induced residual compressive stress. Our this result is consistent with recent clarification on similar issue for cantilever beams with surface tension, which implies that the existing models have incorrectly predicted an invalid tensile pre-stress for an elastic plate with in-plane movable edges which leads to significant overestimation of postbuckling load and free vibration frequencies. In addition, our numerical examples indicated that surface stresses can moderately increase or decrease postbuckling load and free vibration frequency of Kirchhoff nanoplate with all in-plane movable edges, depending on the surface elasticity parameters and the geometrical dimensions of nanoplates.

1. Introduction

Over the last decades, beam- and plate-like elastic nanostructures have been widely used in MEMS/NEMS [1–5]. Owing to the large ratio of surface to volume, the effects of surface tension and surface elasticity on the mechanical behavior of such elastic nanostructures have attracted considerable attention. As experimental and atomistic simulation methods for nanoscale materials are complex, expensive and time-consuming, effective theoretical methods, such as continuum elastic models, have been widely used to investigate the mechanical behavior of elastic nanostructures. In an effort to study surface elasticity of small-scale elastic materials, Gurtin and Murdoch (GM) [6,7] developed a theoretical framework of surface elasticity in the 1970s, and the related surface elasticity parameters were estimated, among others, by Miller and Shenoy [8] using atomistic simulation. In the past decade, the linearized GM model has been widely used to study the influence of surface elasticity on static bending, compressed buckling and vibration of nanobeams, nanofilms or nanoplates [9–14].

The GM model treats the initial surface stress σ_0, called "surface tension", as a finite value, while the surface tension σ_0-induced residual stress is treated as infinitesimal. Consequently, the original form of the GM model has two features: 1) the surface stresses depend not only on surface strains but also on some displacement gradients which cannot be expressed in terms of surface strains; 2) the surface tension σ_0 appears as a coefficient in the linear equation of motion/equilibrium while the σ_0-induced residual stress is absent, which predicts an unbalanced tensile pre-stress for thin nanobeams and nanoplates. Consequently, for instance, for one-dimensional cantilever nanobeams, the original form of GM model could have given a non-zero tensile axial stress caused by σ_0. Actually, Gurtin et al. [15] have correctly stated that the σ_0-induced compressive residual stress must be treated as a finite value and added to eliminate the tensile axial force caused by σ_0, and then there will be no an unbalanced axial force in a cantilever beam. Park and Klein [16] and Yun and Park [17] investigated surface stress effects on vibration and bending behavior of metal nanowires using the surface Cauchy-Born model. Their results indicated that whether a non-zero axial force exists depends on the end conditions. Song et al. [18] used a continuum model to analyze mechanical behaviors of nanowires with surface tension and surface tension-induced residual stresses. By comparing surface Cauchy-Born model

* Corresponding author.
E-mail address: kyxu@shu.edu.cn (K.Y. Xu).

http://dx.doi.org/10.1016/j.ijnonlinmec.2016.10.013
Received 9 June 2016; Received in revised form 26 October 2016; Accepted 26 October 2016
Available online 29 October 2016
0020-7462/

and generalized Young–Laplace model with experimental data, Song et al. [18] concluded that surface tension-induced residual stress is essential for correctly predicting overall mechanical behavior of cantilever nanobeams/nanowires.

Recently, some modified versions of GM model have been proposed by some authors [19–21] to address the above-mentioned issues. In the present paper, a strain-consistent model proposed in Ru [22] is employed to study large deflection behavior of nanoplates with surface tension and surface tension-induced residual stresses. In particular, modified Von Kármán equations are derived for the Kirchhoff nanoplates with surface tension and surface tension-induced residual stresses, and the derived equations are used to study the large deflection mechanical behaviors of plate-like nanostructures.

2. An elastic nanoplate with surface tension and surface elasticity

An elastic isotropic plate of uniform thickness h is considered here. Rectangular Cartesian coordinates (x, y, z) are introduced where the xy-plane coincides with the geometric mid-plane of the plate and the z-coordinate taken positive downward. According to Kirchhoff's hypothesis, the displacement field can be represented by

$$u_1(x, y, z, t) = u^* + u - z\frac{\partial w(x, y, t)}{\partial x}, \quad u_2(x, y, z, t) = v^* + v - z\frac{\partial w(x, y, t)}{\partial y}, \quad u_3(x, y, z, t) = w(x, y, t). \tag{1}$$

where u, v and w denote the displacements of a material point at $(x, y, 0)$ on the mid-plane caused by applied mechanical loadings, and u^* and v^* are the in-plane residual displacements induced by the initial surface stress σ_0.

For the plate with surface tension σ_0 and σ_0-induced residual stresses, the nonlinear strains of von Kármán type are given by

$$\varepsilon_{xx} = \varepsilon_{xx}^* + \varepsilon_{xx}^0 - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \varepsilon_{yy}^* + \varepsilon_{yy}^0 - z\frac{\partial^2 w}{\partial y^2},$$
$$\gamma_{xy} = 2\varepsilon_{xy} = \gamma_{xy}^* + \gamma_{xy}^0 - 2z\frac{\partial^2 w}{\partial x \partial y}, \tag{2}$$

where $(\varepsilon_{xx}^*, \varepsilon_{xx}^*, \gamma_{xy}^*)$ are the in-plane residual strains induced by surface tension σ_0, and

$$\varepsilon_{xx}^0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{yy}^0 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy}^0 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \tag{3}$$

are the in-plane strains of the mid-plane, called "membrane strains". The membrane strains satisfy St. Venant's compatibility condition

$$\frac{\partial^2 \varepsilon_{xx}^0}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{4}$$

Based on Hooke's law, the constitutive equations of the bulk plate can be written as

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) = \frac{E}{1-\nu^2}(\varepsilon_{xx}^* + \nu\varepsilon_{yy}^*) + \frac{E}{1-\nu^2}(\varepsilon_{xx}^0 + \nu\varepsilon_{yy}^0) - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}) = \frac{E}{1-\nu^2}(\varepsilon_{yy}^* + \nu\varepsilon_{xx}^*) + \frac{E}{1-\nu^2}(\varepsilon_{yy}^0 + \nu\varepsilon_{xx}^0) - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right),$$
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}^* + \frac{E}{2(1+\nu)}\gamma_{xy}^0 - \frac{Ez}{1-\nu^2}(1-\nu)\frac{\partial^2 w}{\partial x \partial y}, \tag{5}$$

where E and ν are the elastic modulus and Poisson's ratio of the bulk plate, respectively.

For surface constitutive relations, the surface strain energy adopted in Ru [22] is

$$\gamma = \sigma_0(1 + \varepsilon_{xx}^s + \varepsilon_{yy}^s) + \frac{1}{2}\lambda_0(\varepsilon_{xx}^s + \varepsilon_{yy}^s)^2 + \mu_0\left[(\varepsilon_{xx}^s)^2 + (\varepsilon_{yy}^s)^2 + \frac{1}{4}(\gamma_{xy}^s)^2\right], \tag{6}$$

where $(\varepsilon_{xx}^s, \varepsilon_{xx}^s, \text{and} \gamma_{xy}^s)$ are surface strains given by (2) with $z = \pm h/2$, $\lambda_0 = \lambda_s + \sigma_0$ and $\mu_0 = \mu_s - \sigma_0$ and λ_s and μ_s are two surface elastic constants. Just as for the GM model, the displacements between surface and bulk material are continuous and the surface strains are equal to the values of bulk strains on the surface. It is verified that the surface stresses can be expressed as [22]

$$\sigma_{xx}^s = \sigma_0 + \lambda_0(\varepsilon_{xx}^s + \varepsilon_{yy}^s) + 2\mu_0\varepsilon_{xx}^s = \sigma_0 + (\lambda_0 + 2\mu_0)\varepsilon_{xx} + \lambda_0\varepsilon_{yy}$$
$$= \sigma_0 + [(\lambda_0 + 2\mu_0)\varepsilon_{xx}^* + \lambda_0\varepsilon_{yy}^*] + [(\lambda_0 + 2\mu_0)\varepsilon_{xx}^0 + \lambda_0\varepsilon_{yy}^0] - \left[(\lambda_0 + 2\mu_0)\frac{\partial^2 w}{\partial x^2} + \lambda_0\frac{\partial^2 w}{\partial y^2}\right]z,$$
$$\sigma_{yy}^s = \sigma_0 + \lambda_0(\varepsilon_{xx}^s + \varepsilon_{yy}^s) + 2\mu_0\varepsilon_{yy}^s = \sigma_0 + (\lambda_0 + 2\mu_0)\varepsilon_{yy} + \lambda_0\varepsilon_{xx}$$
$$= \sigma_0 + [(\lambda_0 + 2\mu_0)\varepsilon_{yy}^* + \lambda_0\varepsilon_{xx}^*] + [(\lambda_0 + 2\mu_0)\varepsilon_{yy}^0 + \lambda_0\varepsilon_{xx}^0] - \left[(\lambda_0 + 2\mu_0)\frac{\partial^2 w}{\partial y^2} + \lambda_0\frac{\partial^2 w}{\partial x^2}\right]z,$$
$$\tau_{xy}^s = \mu_0\gamma_{xy}^s = \mu_0\gamma_{xy} = \mu_0\gamma_{xy}^* + \mu_0\gamma_{xy}^0 - 2\mu_0\frac{\partial^2 w}{\partial x \partial y}z. \tag{7}$$

As we known, surface stresses in the original GM model depend on surface strains and some gradients of in-plane displacements (u, v), and they cannot be expressed in terms of surface strains and the deflection w. The present model, as a simplified GM model, makes the surface stresses in Eq. (7) depend on surface strains and the deflection w only, independent of any non-strain gradients of in-plane displacements (u, v). It is this feature that makes it possible to express the von Kármán compatibility Eq. (4) in terms of the stress function φ and deflection w (see Eq. (17)).

Thus, the resultant mid-plane membrane forces are given by

$$N_{xx} = \int_{-h/2}^{h/2}\sigma_{xx}dz + \int_c \sigma_{xx}^s dc = 2\sigma_0 + N_{xx}^* + C_1\varepsilon_{xx}^0 + C_2\varepsilon_{yy}^0,$$
$$N_{yy} = \int_{-h/2}^{h/2}\sigma_{yy}dz + \int_c \sigma_{yy}^s dc = 2\sigma_0 + N_{yy}^* + C_1\varepsilon_{yy}^0 + C_2\varepsilon_{xx}^0,$$
$$N_{xy} = \int_{-h/2}^{h/2}\sigma_{xy}dz + \int_c \tau_{xy}^s dc = N_{xy}^* + C_3\gamma_{xy}^0, \tag{8}$$

$$C_1 = \frac{Eh}{1-\nu^2} + 2\lambda_0 + 4\mu_0, \quad C_2 = \frac{Eh\nu}{1-\nu^2} + 2\lambda_0, \quad C_3 = \frac{Eh}{2(1+\nu)} + 2\mu_0, \tag{9}$$

where c is the perimeter of the cross-section, h is the plate thickness, N_{xx}^*, N_{yy}^*, and N_{xy}^* are the residual mid-plane membrane forces caused by surface tension-induced residual stresses, defined by

$$N_{xx}^* = \frac{Eh}{1-\nu^2}(\varepsilon_{xx}^* + \nu\varepsilon_{yy}^*) + 2[(\lambda_0 + 2\mu_0)\varepsilon_{xx}^* + \lambda_0\varepsilon_{yy}^*] = C_1\varepsilon_{xx}^* + C_2\varepsilon_{yy}^*,$$
$$N_{yy}^* = \frac{Eh}{1-\nu^2}(\varepsilon_{yy}^* + \nu\varepsilon_{xx}^*) + 2[(\lambda_0 + 2\mu_0)\varepsilon_{yy}^* + \lambda_0\varepsilon_{xx}^*] = C_1\varepsilon_{yy}^* + C_2\varepsilon_{xx}^*,$$
$$N_{xy}^* = \left[\frac{E}{2(1+\nu)} + 2\mu_0\right]\gamma_{xy}^* = C_3\gamma_{xy}^*. \tag{10}$$

Thus the membrane forces (8) are given in terms of the in-plane membrane strains (3) and the deflection w. So the in-plane membrane strains given by the simplified GM model can be written in term of in-plane membrane forces as

$$\varepsilon_{xx}^0 = \frac{C_1(N_{xx} - N_{xx}^* - 2\sigma_0) - C_2(N_{yy} - N_{yy}^* - 2\sigma_0)}{C_1^2 - C_2^2},$$
$$\varepsilon_{yy}^0 = \frac{C_1(N_{yy} - N_{yy}^* - 2\sigma_0) - C_2(N_{xx} - N_{xx}^* - 2\sigma_0)}{C_1^2 - C_2^2},$$
$$\gamma_{xy}^0 = \frac{N_{xy} - N_{xy}^*}{C_3}. \tag{11}$$

In addition, the total resultant moments can be expressed as