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Lyapunov exponents in discrete modelling of a cantilever beam impacting on a moving base

Krzysztof Czolczynski[a], Andrzej Okolewski[b], Barbara Blazejczyk-Okolewska[a,*]

[a] Division of Dynamics, Lodz University of Technology, 1/15 Stefanowskiego St., Poland
[b] Institute of Mathematics, Lodz University of Technology, 215 Wolczanska St., Poland

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ABSTRACT

The dynamic behaviour of a cantilever beam of an unnegligible large mass and with a concentrated mass fixed at its end, which impacts on a movable base according to Hertz's damp law, is studied. A new finite element reference model of the system and its lower-dimensional substitutive models with one degree or two degrees of freedom are developed. The qualitative-type as well as quantitative-type applicability limits of these substitutive models are discussed - the latter ones are described in terms of the corresponding spectra of Lyapunov exponents.

1. Introduction

Vibrations of mechanical systems with impacts have been extensively studied starting from pioneering works by Goldsmith [1], Feigin [2], Peterka [3] and Fillippov [4]. The reason of this interest lies in the fact that such motions are a typical feature of various engineering applications (e.g., see Ref. [5] and the references therein). Due to a strongly nonlinear nature of the collision process, vibro-impact systems can exhibit very diverse dynamic behaviours like chaotic motion, intermittency, Devil's attractors, a Feigenbaum scenario and different types of grazing bifurcations (e.g., [6–14]).

In most of the investigations carried out so far either physical models with a finite number of degrees of freedom, composed of rigid and heavy masses connected by a massless spring, or physical models of continuous elastic systems, that is, systems with a considerable, unnegligible mass of elastic elements, have been employed. These systems have been intensively studied both in the case of rigid (e.g., [9,10,16–24]) as well as soft amplitude constraints (e.g., [12,13,17–21,24–41]), and the r*esearchers' attention has been* mostly focused on periodic solution stability, bifurcations and singularities in vibro-impact dynamics. A finite-dimensional nonlinear model of a cantilever beam with a tip mass that is impacted by a shaker was developed and thoroughly studied by Balachandran [42], Long et al. [43] and Dick et al. [44]. In Ref. [15], continuous elastic systems that have rigid concentrated masses impacting against an unmovable non-deformable base have been investigated, a way to determine substitutive models with one degree and two degrees of freedom against a finite element model has been proposed and qualitative-type applicability limits of these substitutive models have been discussed.

In this paper, we deal with a cantilever beam with an unnegligible large mass whose one end is harmonically kinetically forced, whereas a concentrated mass is located on its second end (in contrast to [15]) and it can impact against a moving damped Hertz-type base (for Hertz's damp contact model, see, e.g., [13,32,33]). The main objective is to develop a new discrete model of this system with impacts and examine the applicability limits of the model with one degree or two degrees of freedom as a substitutive system for such a reference model. To attain this, the substitutive systems have been proposed, the spectra of Lyapunov exponents have been determined for the systems under consideration, a certain numerical characteristics of dynamic behaviours of the system related to energy dissipation caused by impacts has been defined with the spectrum of Lyapunov exponents (see Section 3) and the values of these characteristics determined for the corresponding systems have been compared. The approach applied has allowed for finding not only qualitative-type applicability limits of the substitutive systems as in Ref. [15], but more restrictive applicability limits of the quantitative type as well. The coefficient proposed by us can be the basis for an introduction of global – for the given range of the control parameter - measures of distances between dynamic behaviours of various systems. To do it, it is enough to count the distances of the corresponding diagrams of this coefficient for various systems applying standard distance measures for real functions, for instance the Kolmogorov distance or the distance in space ℓ^p for $p \geq 1$.

In Section 2, a physical model shown in Fig. 1 with the finite element method, hereafter referred to as the FEM model (Fig. 2a), and models of systems with one degree or two degrees of freedom (here-

* Corresponding author.
E-mail address: okolbar@p.lodz.pl (B. Blazejczyk-Okolewska).

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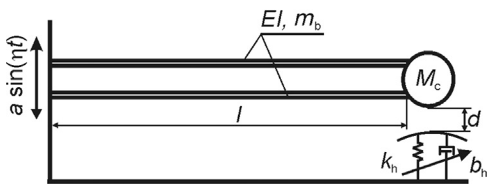

Fig. 1. Cantilever beam with impacts under consideration.

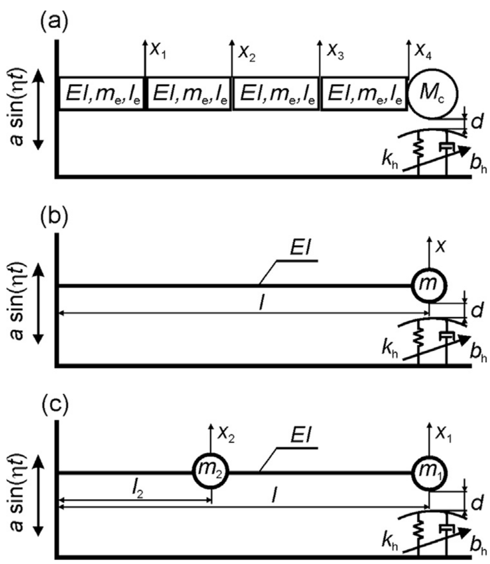

Fig. 2. Equivalent models of the cantilever beam; FEM model (a), 1DOF model (b); 2DOF model (c).

after referred to as the 1DOF model (Fig. 2b) or the 2DOF model (Fig. 2c)) are developed. Various dynamic characteristics (bifurcation diagrams of displacement, maximal Lyapunov exponents and the coefficient of dynamic behaviours based on the spectrum of Lyapunov exponents) determined for the FEM model are compared to the respective characteristics for the substitutive systems, that is, for the 1DOF model and the 2DOF model in Section 3. On the basis of these comparisons, a conclusion can be drawn that substitutive models with a low number of degrees of freedom can be applied to model and analyse the motion of elastic beams of a considerable mass and having an additional concentrated mass that impacts against a moving base.

2. Discrete models of the system

Consider a simple cantilever beam system depicted in Fig. 1, which comprises a concentrated mass M_c and two leaf springs of a length l, a mass m_b and a bending stiffness EI, connecting this concentrated mass with the base subject to kinematic excitation of an amplitude a and a frequency η. The boundary conditions prevent rotation of the mass M_c.

2.1. FEM model

Fig. 2a depicts a mathematical model of the system under investigation, generated with the finite element method. It was found in some simple numerical experiments that four finite elements were enough to model vibrations of the cantilever beam: the values of the first three resonance frequencies of this model (without a concentrated mass) in the experiment corresponded to the results of analytical calculations with accuracy up to three significant digits. The mass M_c located at one end of the beam has a fender with which it can impact on the moving base. If the system is in the static equilibrium position, the bottom part of the mass M_c (fender) is situated at a distance d from the base.

The matrix equation of the system motion can be written in the following form:

$$[M]\{\ddot{x}\} + [T]\{\dot{x}\} + [K]\{x\} + \{F_d\} = \{F\}a\sin(\eta t). \tag{1}$$

The stiffness matrix $[K]$ takes the form:

$$[K] = \begin{bmatrix} k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 & 0 \\ k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 & 0 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & k_{34}^2 + k_{12}^3 & k_{13}^3 & k_{14}^3 & 0 \\ k_{41}^2 & k_{42}^2 & k_{43}^2 + k_{21}^3 & k_{44}^2 + k_{22}^3 & k_{23}^3 & k_{24}^3 & 0 \\ 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3 + k_{11}^4 & k_{34}^3 + k_{12}^4 & k_{13}^4 \\ 0 & 0 & k_{41}^3 & k_{42}^3 & k_{43}^3 + k_{21}^4 & k_{44}^3 + k_{22}^4 & k_{23}^4 \\ 0 & 0 & 0 & 0 & k_{31}^4 & k_{32}^4 & k_{33}^4 \end{bmatrix}. \tag{2}$$

Its components are the stiffness matrices $[k_{rs}^1]_{3\le r,s\le 4}$, $[K^2]$, $[K^3]$ and $[k_{rs}^4]_{1\le r,s\le 3}$ of the first, second, third and fourth finite element, respectively, where:

$$[K^i] = \begin{bmatrix} k_{11}^i & k_{12}^i & k_{13}^i & k_{14}^i \\ k_{21}^i & k_{22}^i & k_{23}^i & k_{24}^i \\ k_{31}^i & k_{32}^i & k_{33}^i & k_{34}^i \\ k_{41}^i & k_{42}^i & k_{43}^i & k_{44}^i \end{bmatrix} = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}, \tag{3}$$

in which E denotes the Young modulus, I is a moment of inertia of the beam cross-section, and l_e stands for individual finite element lengths.

The inertia matrix takes the form:

$$[M] = \begin{bmatrix} m_{33}^1 + m_{11}^2 & m_{34}^1 + m_{12}^2 & m_{13}^2 & m_{14}^2 & 0 & 0 & 0 \\ m_{43}^1 + m_{21}^2 & m_{44}^1 + m_{22}^2 & m_{23}^2 & m_{24}^2 & 0 & 0 & 0 \\ m_{31}^2 & m_{32}^2 & m_{33}^2 + m_{11}^3 & m_{34}^2 + m_{12}^3 & m_{13}^3 & m_{14}^3 & 0 \\ m_{41}^2 & m_{42}^2 & m_{43}^2 + m_{21}^3 & m_{44}^2 + m_{22}^3 & m_{23}^3 & m_{24}^3 & 0 \\ 0 & 0 & m_{31}^3 & m_{32}^3 & m_{33}^3 + m_{11}^4 & m_{34}^3 + m_{12}^4 & m_{13}^4 \\ 0 & 0 & m_{41}^3 & m_{42}^3 & m_{43}^3 + m_{21}^4 & m_{44}^3 + m_{22}^4 & m_{23}^4 \\ 0 & 0 & 0 & 0 & m_{31}^4 & m_{32}^4 & m_{33}^4 + M_c \end{bmatrix}. \tag{4}$$

This matrix consists of the inertia matrices $[m_{rs}^1]_{3\le r,s\le 4}$,$[M^2]$, $[M^3]$ and $[m_{rs}^4]_{1\le r,s\le 3}$ of the first, second, third and fourth finite element, respectively, in which

$$[M^i] = \begin{bmatrix} m_{11}^i & m_{12}^i & m_{13}^i & m_{14}^i \\ m_{21}^i & m_{22}^i & m_{23}^i & m_{24}^i \\ m_{31}^i & m_{32}^i & m_{33}^i & m_{34}^i \\ m_{41}^i & m_{42}^i & m_{43}^i & m_{44}^i \end{bmatrix} = \frac{m_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix} \tag{5}$$

with m_e standing for individual finite element masses.

In the investigations discussed here, damping was restricted to external damping. This means that the damping matrix is proportional to the inertia matrix (see, e.g., [45]):

$$[T] = \upsilon[M]. \tag{6}$$

Here υ denotes a coefficient of external damping.

The structure of vectors of displacements and forces is as follows:

$$\{x\} = \begin{Bmatrix} x_1 \\ \varphi_1 \\ x_2 \\ \varphi_2 \\ x_3 \\ \varphi_3 \\ x_4 \end{Bmatrix}; \qquad \{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{7}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EI}{l_e^3}\begin{Bmatrix} 12 \\ -6l_e \end{Bmatrix} + \frac{\eta^2 m_e}{420} \times \begin{Bmatrix} 54 \\ -13l_e \end{Bmatrix}.$$

An impact of the mass M_c on the base was modelled with Hertz's law (see, e.g., [13,33]):