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Impinging rotational stagnation-point flows

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ABSTRACT

A rotational stagnation-point flow of fluid density ρ_1 and kinematic viscosity ν_1 impinges normal to another rotational stagnation-point flow of fluid density ρ_2 and viscosity ν_2. Results are compared with a previous study on the normal impingement of two Homann stagnation-point flows for which the flow in the far field is irrotational.

1. Introduction

Axisymmetric stagnation-point flow on a flat surface was first considered by Homann [1]. Many variations on that problem have been studied including the effects of wall stretching, and suction and blowing through a porous wall; see the review by Wang [2]. An axisymmetric stagnation-point flow was discovered by Agrawal [3]. In contrast to the irrotational outer flow of Homann [1], this flow is rotational in the far field. Agrawal [3] derived his solution using spherical coordinates. The nature of the flow becomes most apparent when cylindrical coordinates (r, z) with coordinate velocities (u, w) are used which furnish the pleasingly simple solution

$$u(r, z) = a\, r\, z, \quad w(r, z) = -a\, z^2 \tag{1.1}$$

in which the parameter a having units $(LT)^{-1}$ measures the strength of the stagnation flow. Clearly the impermeable and no-slip conditions are satisfied at the surface z=0.

The present study is the fourth in a series of papers devoted to extensions of this seldom studied stagnation-point flow. Weidman [4] considered Agrawal stagnation-point flow impinging normal to a rotating plate, analogous to the problem studied by Hannah [5] who considered the same situation but with Homann stagnation-point flow. In a sequel Weidman [6] studied Agrawal stagnation-point flow impinging normal to a radially stretching plate; this represents a variation of the Homann stagnation-point flow impinging normal to a radially stretching plate studied by Mahapatra and Gupta [7]. In a third paper on this subject, Weidman [8] investigated Agrawal stagnation-point flow impinging normal to a flat quiescent liquid surface.

It should be mentioned that other rotational stagnation-point flows exist. The rotational aspect in the far field is obtained by superposition of a sideways shear flow onto a normal stagnation-point flow. For two-dimensional rotational stagnation-point flows see the one-fluid studies of Stuart [9], Tamada [10] and Dorrepaal [11] and the two-fluid study of Tilley and Weidman [12]. For an axisymmetric rotational stagnation-point flow along the surface of a cylinder, see Okamoto [13] and Weidman and Putkaradze [14,15].

In the present study we consider Agrawal stagnation-point flow of one fluid impinging normal to an Agrawal stagnation-point flow of another fluid. This is analogous to the problem of normally impinging Homann stagnation-point flows studied by Wang [16]. All the above studies represent exact solutions of the Navier–Stokes equation in the manner defined in Drazin and Riley [17]. While the Homann [1] flow is irrotational in the far field the Agrawal [3] flow is not. It should be here noted that Davey [18] studied the rotational flow near a forward stagnation point, but his analysis is restricted to the boundary-layer approximation and does not represent an exact solution of the Navier–Stokes equations.

The presentation is as follows. The exact similarity reduction of the Navier–Stokes equation is given in Section 2 along with introductory figures showing the region of acceptable solutions. Further results of numerical calculations are given in Section 3 and the paper terminates with a discussion and concluding remarks in Section 4.

2. Problem formulation

The swirl-free axisymmetric problem is formulated using cylindrical coordinates (r,z) with corresponding velocities (u,w). Following Wang [16], we denote velocities in the upper-half plane by $u_1(r, z_1)$, $w_1(z_1)$ with z_1 pointing upward and velocities in the lower-half plane by $u_2(r, z_2)$, $w_2(z_2)$ with z_2 pointing downward, for which the horizontal interface between the stagnation-point flows lies at $z_1 = z_2 = 0$. A sketch of the system is presented in Fig. 1. A solution form that gives rise to axisymmetric rotational Argawal stagnation-point in the far field above the interface, and satisfies the equation of continuity for incompressible flow, is known to be of the form [4,6,8]

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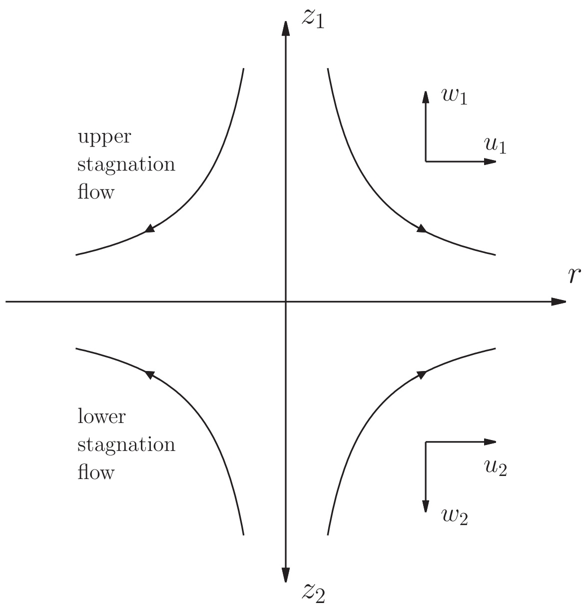

Fig. 1. Sketch of the Agrawal stagnation-point flow in the upper layer impinging on an Agrawal stagnation-point flow in the lower layer showing the coordinate and velocity system employed.

$$u_1(r,\eta) = a_1^{2/3}\nu_1^{1/3} r F'(\eta), \quad w_1(\eta) = -2a_1^{1/3}\nu_1^{2/3}F(\eta), \quad \eta = \left(\frac{a_1}{\nu_1}\right)^{1/3} z_1. \tag{2.1}$$

where a prime denotes differentiation with respect to η. Inserting this *ansatz* into the Navier–Stokes equations furnishes the boundary-value problem for $\eta \ge 0$ as

$$F''' + 2FF'' - F'^2 = 0, \quad F(0) = 0, \quad F'(0) = \alpha, \quad F''(\infty) = 2 \tag{2.2}$$

where α is a dimensionless parameter measuring the local interfacial velocity which may be chosen at will.

For the lower fluid the same similarity solution form applies, *viz.*

$$u_2(r,\zeta) = a_2^{2/3}\nu_2^{1/3} r G'(\zeta), \quad w_2(\eta) = -2a_2^{1/3}\nu_2^{2/3}G(\zeta), \quad \zeta = \left(\frac{a_2}{\nu_2}\right)^{1/3} z_2 \tag{2.3}$$

where now a prime denotes differentiation with respect to ζ. Insertion into the Navier–Stokes equations for the lower region $\zeta \ge 0$ furnishes the boundary-value problem

$$G''' + 2GG'' - G'^2 = 0, \quad G(0) = 0, \quad G'(0) = \beta, \quad G''(\infty) = 2 \tag{2.4}$$

where β is a dimensionless parameter measuring the local interfacial velocity. For fixed $\alpha > 0$ one finds β by matching dimensional radial velocities at the interface.

Integrating the z-component of the Navier–Stokes equation in the upper layer gives the pressure field

$$p_1(\eta) = p_0 - 2a_1^{2/3}\rho_1\nu_1^{4/3}(F^2(\eta) + F'(\eta) - \alpha) \tag{2.5a}$$

and doing the same for the lower layer, matching to the upper layer pressure at the interface, gives

$$p_2(\zeta) = p_0 - 2a_2^{2/3}\rho_2\nu_2^{4/3}\beta(G^2(\zeta) + G'(\zeta) - \beta) \tag{2.5b}$$

in which p_0 is the stagnation pressure. Since there is no radial variation of the pressure, there is no deflection of the free surface as in the problem studied by Wang [16]. Wang's results depend on the flat interface criterion

$$\frac{\delta}{r} = \frac{A^2 r}{2(\rho_2/\rho_1 - 1)g} \ll 1 \tag{2.6}$$

where δ is the deflection of the free surface, A is the strain rate of the Homann stagnation flow with units (T^{-1}), and g is gravity.

In the sequel the upper layer fluid parameters ρ_1 and ν_1 are considered known as is the strength a_1 of the upper rotational stagnation-point flow. The matching conditions at the interface require equal radial shear stresses and velocities. Matching stresses at the interface

$$\rho_1\nu_1 \left.\frac{\partial u_1}{\partial z_1}\right|_{z_1=0} = -\rho_2\nu_2 \left.\frac{\partial u_2}{\partial z_2}\right|_{z_2=0} \tag{2.7a}$$

furnishes the relation

$$\left(\frac{a_2}{a_1}\right)\left(\frac{\nu_2}{\nu_1}\right)\left(\frac{\rho_2}{\rho_1}\right) = -\chi, \quad \text{where} \quad \chi \equiv \frac{F''(0)}{G''(0)}. \tag{2.7b}$$

Matching the velocities gives

$$a_1^{2/3}\nu_1^{1/3}\alpha = a_2^{2/3}\nu_2^{1/3}\beta. \tag{2.8}$$

Consider first the case of an immobile interface $\alpha = \beta = 0$ for which (2.8) is satisfied identically. Then the solutions for $F(\eta)$ and $G(\zeta)$ are each that of Agrawal stagnation-point flow, *viz.*

$$F(\eta) = \eta^2, \qquad G(\zeta) = \zeta^2 \tag{2.9}$$

for which $F''(0) = G''(0) = 2$ and hence from (2.7b)

$$\left(\frac{a_2}{a_1}\right)\left(\frac{\nu_2}{\nu_1}\right)\left(\frac{\rho_2}{\rho_1}\right) = -1. \tag{2.10}$$

Since all ratios on the left are positive, an immobile interface is not possible. In fact, from (2.7b) it is clear that one must have $\chi < 0$ and we choose to satisfy this by orchestrating $F''(0)$ to be negative. As shown in Weidman [8], $F''(0)$ first becomes negative when $\alpha = \alpha_0 = 1.77304$; in that problem for Agrawal stagnation-point flow impinging on a quiescent liquid surface, solutions are available for $\alpha < \alpha_0$ only. Here, however, we need $\alpha > \alpha_0$ and a plot of $F''(0)$ up to $\alpha = 3.0$ is shown in Fig. 2; the value $F''(0) = 2$ at $\alpha = 0$ is the Agrawal stress and the dot corresponds to zero upper liquid interfacial stress at $\alpha = \alpha_0$.

An inequality exists for the radial stresses to match at the interface. For our choice $F''(0) < 0$ one must have $\alpha > \alpha_0$ and then for $G''(0) > 0$ one must have $0 < \beta < \alpha_0$. Thus

$$\sigma \equiv \frac{\alpha}{\beta} = \left(\frac{a_2}{a_1}\right)^{2/3}\left(\frac{\nu_2}{\nu_1}\right)^{1/3} > 1 \tag{2.11a}$$

from which one finds the necessary inequality

$$\left(\frac{a_2}{a_1}\right) > \left(\frac{\nu_2}{\nu_1}\right)^{-1/2}. \tag{2.11b}$$

A plot of the region of solutions $\sigma > 1$ is shown in FIg. 3.

In the present study, owing to the fact that the interfacial pressure

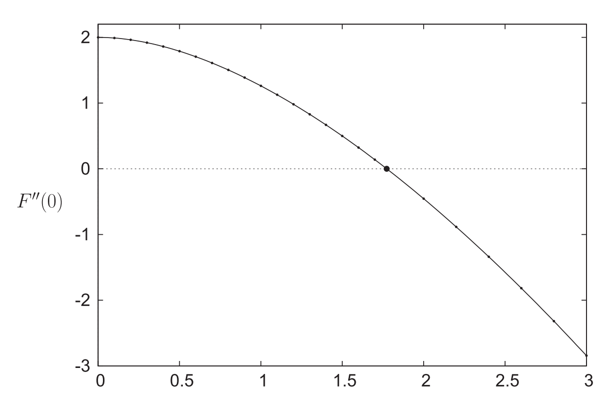

Fig. 2. Radial interface stress parameter $F''(0)$ for Agrawal stagnation-point flow as a function of the interface velocity parameter α. The dot at $\alpha = \alpha_0 = 1.77304$ represents the point of zero radial shear stress.