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Optimal systems of Lie subalgebras for a two-phase mass flow

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A R T I C L E I N F O

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A B S T R A C T

We apply the Lie symmetry method to a two-phase mass flow model (Pudasaini, 2012 [18]) and construct one-, two- and three-dimensional optimal systems of Lie subalgebras corresponding to the non-linear PDEs. As an optimal system contains structurally important information about different types of invariant solutions, it provides precise insights into all possible invariant solutions emerging from infinitesimal symmetries. We use the optimal system of one-dimensional Lie subalgebras to reduce the two-phase mass flow model to other systems of PDEs. Using the fact that the Lie bracket contains information about further reduction, we further reduce to systems of ODEs and PDEs. We solve a system numerically and present results for different physical and Lie parameters. Simulations reveal fluid and solid dynamics are distinctly sensitive to different Lie parameters, whereas both phases are influenced by the solid and the fluid pressure parameters. Higher pressure gradients result in higher flow velocities and lower flow heights. Fluid velocities dominate solid velocities, but the solid heights are higher than the fluid heights. Results provide an overall picture of the physical process, and the coupled dynamics of the solid and fluid phase velocities and the flow heights. These are physically meaningful results in sheared inclined channel flow of coupled two-phase mixture. This confirms the consistency of the obtained similarity solutions and potential applicability of the models and the constructed optimal systems.

1. Introduction

Rapid gravity mass flows and gravity currents such as debris flows, debris floods, and water waves are common natural phenomena. Due to their complex nature these events pose greater challenges to environmental, geophysical and engineering communities [4–7,10,16,17,21,22,31]. In this paper, we are mainly concerned with debris flows, which are effectively two-phase gravity-driven mass flows consisting of a mixture of solid particles and viscous fluid. These flows are mainly described by the relative motion between the solid and the fluid phases. Their evolution primarily depends on the solid–fluid mixture composition, mixture interactions, and mixture dynamics modelled by the driving forces. The debris flows are extremely destructive natural hazards. Hence, we need a reliable way to predict the dynamics of the flow. There has been extensive (field, experimental, theoretical, and numerical) studies investigating the dynamics, and consequences of these flows including their industrial applications [9,21,25,26,28]. In [18] Pudasaini presented a new theory, which describes the different interactions between the solid and the fluid in two-phase debris flows including several fundamentally important and dominant physical aspects which constitutes the most generalized two-phase flow model, as a set of partial differential equations in the conservative form [18,19,21] available to date. This theory supports the strong interactions between the solid and the fluid phases.

In this paper, we apply the Lie symmetry method to the two-phase mass flow model [18] and construct optimal systems of subalgebras corresponding to this system of non-linear PDEs. Lie symmetry method is an important tool for examining differential equations and constructing their solutions [1,8]. The heart of this method lies in the construction of the invariant solutions, the solutions which are invariant under the Lie subgroups of the Lie group of symmetries, and the construction of new solutions from known solutions by using symmetries. Invariant solutions are constructed by reducing the given system of PDEs into another system of PDEs or ODEs with the aid of Lie subgroups and solving the reduced system. Since almost every Lie group of symmetries has infinitely many Lie subgroups, it seems the quest for invariant solutions is a very challenging task. As a symmetry transformation maps one solution to another, practically, it is sufficient to find the invariant solutions which are not related by any member of the Lie group of symmetries. This suggests to look for ways to

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determine Lie subgroups which generates fundamentally distinct invariant solutions. It is of great importance from the mathematical point of view as well as constraining the physical and engineering applications, as it provides precise insights into all possible invariant solutions of the system. This problem can be solved by classifying the Lie subgroups of the Lie group of symmetries, often called the optimal system of s-parameter subgroups [11,12]. It possesses an infinitesimal counterpart called the optimal system of s-parameter subalgebras.

In the literature, mostly, single-phase, inviscid and pressure-driven shallow water flows in horizontal channels with no friction models are investigated. For example, in [15] similarity forms for two-layer shallow water equations are derived and discussed where the coupling is via the reduced gravity parameter. In [1], GhoshHajra et al. discuss a two-phase mass flow model whose driving forces consist of gravity, buoyancy, and hydraulic pressure gradients for the solid and fluid phases respectively. The two-phase mass flow model utilized there is derived from a general two-phase mass flow model [18] as a mixture of sediment particles and viscous fluid with strong non-linear phase interactions. These interactions pose great challenges in constructing exact solutions as compared to the effectively single-phase gravity mass flows. Thus, the problem, which is the study of the phase interaction between the solid and the fluid in a two-phase mass flow, is highly non-trivial and novel.

Using an elementary Lie transformation of the model [18], GhoshHajra et al. [1] constructed several similarity forms and similarity variables, which generalizes several similarity variables and similarity forms obtained in the literature by using Lie symmetry group transformations [2,3,13–15,23,24,27,30]. They also constructed several reduced homogeneous and non-homogeneous systems of ODEs and provided some analytical and numerical solutions. Here, we consider the simplified model from [1] and advance further by constructing their optimal systems of Lie subalgebras and providing simulations and applications. For this, we first briefly discuss the two-phase mass flow model and the symmetry Lie algebra of the given system of PDEs from [1]. Then, we construct the optimal system of Lie subalgebras of dimensions one, two, and three. Further, we analyze in detail the reduced system of PDEs obtained using the one-dimensional Lie subalgebras. We construct several reduced systems of ODEs and PDEs. We also present simulation results for some systems in order to have an in-depth and quantitative analyses of these systems. This allows for the analysis of impact of the model parameters in the physical–mathematical model [18] and the Lie parameters in the optimal system under consideration. As all the reduced ODE systems obtained here are non-homogeneous, these systems are more complex both from the mathematical structure and the underlying physics than the systems presented and discussed in [1]. Consequently, the new systems potentially cover wider spectrum of applications.

2. The two-phase mass flow model and the symmetry Lie algebra

We consider the general two-phase debris flow model [18] that was reduced to one-dimensional inclined channel flow [21]. For completeness, we briefly recall the basic features of the model equations [1].[1] The depth-averaged mass and momentum conservation equations for the solid and fluid phases are as follows [18]:

$$\frac{\partial h_s}{\partial t} + \frac{\partial Q_s}{\partial X} = 0, \quad \frac{\partial h_f}{\partial t} + \frac{\partial Q_f}{\partial X} = 0, \tag{1}$$

$$\frac{\partial Q_s}{\partial t} + \frac{\partial}{\partial X}(Q_s^2 h_s^{-1}) + \frac{\partial}{\partial X}\left(\frac{\beta_s}{2} h_s(h_s + h_f)\right) = h_s S_s,$$
$$\frac{\partial Q_f}{\partial t} + \frac{\partial}{\partial X}(Q_f^2 h_f^{-1}) + \frac{\partial}{\partial X}\left(\frac{\beta_f}{2} h_f(h_f + h_s)\right) = h_f S_f, \tag{2}$$

where S_s, S_f are the solid and fluid net driving forces, given by $S_s = \sin\zeta - \tan\delta(1-\gamma)\cos\zeta$, $S_f = \sin\zeta$. The dynamical variables and parameters are

$$h_s = \alpha_s h, \ h_f = \alpha_f h; \ Q_s = h_s u_s = \alpha_s h u_s, \ Q_f = h_f u_f = \alpha_f h u_f; \ \beta_s = \varepsilon K p_{b_s},$$
$$\beta_f = \varepsilon p_{b_f}, \ p_{b_f} = \cos\zeta, \ p_{b_s} = (1-\gamma) p_{b_f}, \ \alpha_f = 1 - \alpha_s, \ \gamma = \frac{\rho_f}{\rho_s}.$$

Here, t is the time, X and Z are coordinates along and normal to the slope with inclination ζ. The solid particles and fluid constituents are denoted by the suffices s and f respectively. The mixture flow depth is h, and the solid and fluid velocities are u_s and u_f respectively. The densities and volume fractions are ρ_s, ρ_f, and α_s, α_f respectively, L and H denote the typical length and depth of the flow with the aspect ratio $\varepsilon = H/L$. Both K, the earth pressure coefficient, and $\tan\delta$, where δ is the friction angle, include frictional behavior of the solid-phase. Here p_{b_f} and p_{b_s} are associated with the effective basal fluid and solid pressures, β_s, β_f are the hydraulic pressure parameters associated with the solid- and the fluid-phases respectively, and γ is the density ratio, $(1-\gamma)$ indicates the buoyancy reduced solid normal load. The solid and fluid fractions in the mixture are h_s, h_f, and the solid and fluid fluxes are Q_s, Q_f.

For convenience, GhoshHajra et al. [1] introduced the transformation

$$x = X - S_s t^2/2, \quad y = X - S_f t^2/2, \ \widehat{Q}_s = Q_s - h_s S_s t, \quad \widehat{Q}_f = Q_f - h_f S_f t. \tag{3}$$

Here, x and y are the moving spatial coordinates for the solid and fluid respectively. Since S_s and S_f are independent, x and y are considered as independent variables [1]. Using (3), Eqs. (1)–(2) become a homogeneous system of partial differential equation:

$$\frac{\partial h_s}{\partial t} + \frac{\partial \widehat{Q}_s}{\partial x} = 0, \ \frac{\partial h_f}{\partial t} + \frac{\partial \widehat{Q}_f}{\partial y} = 0, \tag{4}$$

$$\frac{\partial \widehat{Q}_s}{\partial t} + \frac{\partial}{\partial x}(\widehat{Q}_s^2 h_s^{-1}) + \frac{\partial}{\partial x}\left(\frac{\beta_s}{2} h_s(h_s + h_f)\right) = 0,$$
$$\frac{\partial \widehat{Q}_f}{\partial t} + \frac{\partial}{\partial y}(\widehat{Q}_f^2 h_f^{-1}) + \frac{\partial}{\partial y}\left(\frac{\beta_f}{2} h_f(h_f + h_s)\right) = 0. \tag{5}$$

Replacing $\widehat{Q}_s = \hat{u}_s h_s$, $\widehat{Q}_f = \hat{u}_f h_f$, introducing the suffices $1{:=}s$, $2{:=}f$ for the variables and parameters associated with the solid and fluid components respectively, and dropping the hats, (4)–(5) changes to

$$\frac{\partial h_1}{\partial t} + \frac{\partial (u_1 h_1)}{\partial x} = 0, \tag{6}$$
$$\frac{\partial h_2}{\partial t} + \frac{\partial (u_2 h_2)}{\partial y} = 0,$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + \beta_1\frac{\partial h_1}{\partial x} + \frac{\beta_1}{2}\frac{\partial h_2}{\partial x} + \frac{\beta_1}{2}\frac{h_2}{h_1}\frac{\partial h_1}{\partial x} = 0,$$
$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial y} + \beta_2\frac{\partial h_2}{\partial y} + \frac{\beta_2}{2}\frac{\partial h_1}{\partial y} + \frac{\beta_2}{2}\frac{h_1}{h_2}\frac{\partial h_2}{\partial y} = 0. \tag{7}$$

As discussed in [1], the mass flow model (6)–(7) which is deduced from the mixture model [18] is different from the other existing models. The fourth and fifth terms associated with $\beta/2$ in (7) that emerge from the pressure-gradients, include buoyancy (through $1-\gamma$), friction (through K and $\tan\delta$), net driving forces (S_s, S_f) and the coordinate transformations (3) that incorporate gravity, friction, and buoyancy. These forces have mechanical significance in explaining the physics of the two-phase gravity mass flows that are not discussed in previous models, and in

[1] This discussion follows [1, pp. 326–327].