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A parametrically excited pendulum with irrational nonlinearity

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ABSTRACT

In this paper, we propose a parametrically excited pendulum with irrational nonlinearity which comprises a simple pendulum linked by a linear spring under base excitation. This parametric vibration system exhibits bistable state and discontinuous characteristics due to the geometry configuration. For small oscillations, this system can be described by Mathieu equation coupled with SD (Smooth and Discontinuous) oscillator whose dynamic response is examined analytically by using the averaging method in both smooth and discontinuous case. Numerical simulations are carried out to demonstrate the complicated dynamic behavior of multiple periodic motions and different types of chaotic motions.

1. Introduction

It is well known that the parametrically excited pendulum has received a great deal of attention due to its rich dynamic behavior from multiple periodic motion to complex chaos, particularly as a typical example of the parametric vibration system, has been the subject of study for a considerable amount of time. Over the past half century, the parametrically excited pendulum has been extensively reported in term of the analytical study, numerical simulation and experimental verification. In the 80's, Leven and Koch et al. examined the chaotic behavior [1] of the parametrically excited pendulum system by means of numerical solution, and then identified the boundaries of subharmonic and homoclinic bifurcations by applying Melnikov and averaging method [2]. In addition, they made an experimental study to verify the periodic and chaotic motions of the parametrically excited pendulum system [3,4]. In the 90's, Clifford and Bishop et al. studied the escape zone [5], classification of rotating periodic orbits [6], locating oscillatory orbits [7] and inverted dynamics [8] of the parametrically excited pendulum system. Early this century, Szrmplińska-Stupnicka employed computer aided methods to examine the criteria for occurrence of transient tumbling chaos in the parametrically driven pendulum [9]. In 2003, Garira. et al extended the classification of rotating solutions identifying new types together with previously known ones [10], as purely rotating, oscilating rotating, straddling rotating and large amplitude rotating. In 2007, Xu and Wiercigroch obtained the approximate analytical solutions of oscillatory and rotational motions by using first order perturbation method [11,12], which has been recently expanded to the higher order terms and extensively studies by Lenci et al. [13]. Particularly, a new concept of using mechanical pendulum systems for wave energy extraction has been given in [14], which is a innovative application of the parametrically excited pendulum. Litak et al. proposed a classification of the complex responses of the parametrically excited pendulum by the recurrence plots [15] in 2010. In addition, an asymmetrically supported inverted pendulum with base excitation was proposed in [16], whose transition curves are numerically and theoretically studied by means of an asymmetric Mathieu equation.

Recently, a rotating pendulum linked by a linear spring with a fixed end [17] is presented, whose chaotic boundaries for the smooth case with a pair of double homoclinic orbits [18] and the discontinuous case with a pair of double heteroclinic-like orbits [19] are obtained by using semi-analytical method. The aim of this paper is to start exploring this rotating pendulum excited by a vertical harmonic oscillation, which differs from the parametrically excited pendulum by the fact that it can bear strongly irrational nonlinearity of multiple bistable and discontinuous characteristics owing to the influence of the coupling of the pendulum and spring.

The main motivation of this paper is to present a novel parametrically excited pendulum with irrational nonlinearity, which exhibits multiple bistable and discontinuous characteristics due to the geometry configuration. The other motivation is to reveal the nature of irrational nonlinearity from small oscillation to large rotation and then to chaotic motion by using analytical investigation and numerical verification. In terms of practical application, this pendulum system can be considered

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as an energy collection device for harnessing tidal power [20,21] via the multiple bistable and rotational motions.

This paper is organized as follows. In Section 2, the equation of motion for the novel parametrically excited pendulum is derived, which is a parametric vibration system with irrational nonlinearity. In Section 3, the unperturbed dynamics of this pendulum system with irrational terms is analyzed directly without using Taylor expansion. In the following section, Section 4, the dynamic response of small angle oscillation is explored by means of the averaging method. While in Section 5, numerical simulations are carried out to demonstrate multiple periodic motions and different types of chaotic motions. Finally we summarize the conclusions and provide the further challenge.

2. Proposed model and the equations

Physical model of the novel parametrically excited pendulum is shown in Fig. 1, which consists of a rotating pendulum linked by a linear spring under a base excitation. We first describe this physical model: a simple pendulum is considered to move in a vertical plane. It is hanged on a rigid shaft and connected by a linear spring, where both the rigid shaft and the spring are fixed on a base. The pendulum, linear spring and the rigid shaft as a whole is subjected to harmonic excitation in vertical direction provided by the vibration base.

Based upon the cartesian coordinates (X,Y) in Fig. 1, we establish the dynamic equation of the novel parametrically excited pendulum by applying Lagrange's equation. The position coordinates of the pendulum ball can be denoted as $(X, Y) = (L\sin x, L\cos x + P)$, and then its velocity ν can be obtained by taking the first derivative of its position coordinates

$$\nu = \sqrt{(X')^2 + (Y')^2} = \sqrt{(Lx')^2 - 2P'Lx'\sin x + (P')^2}. \tag{1}$$

The kinetic energy T, potential energy V and dissipative energy Ψ of the novel parametrically excited pendulum system can be written as

$$T = \frac{1}{2}m\nu^2 = \frac{1}{2}m((Lx')^2 - 2P'Lx'\sin x + (P')^2),\ V = -mg(L\cos x + P) + \frac{1}{2}k\left(\sqrt{L^2 + h^2 - 2Lh\cos x} - l\right)^2,\ \Psi = \frac{1}{2}C(Lx')^2, \tag{2}$$

Substituting $L = T - V$ and Ψ into the following Lagrange's equation

$$\frac{d}{d\bar{t}}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} + \frac{\partial \Psi}{\partial x'} = 0, \tag{3}$$

the equation of the novel parametrically excited pendulum system can be derived and written as follows

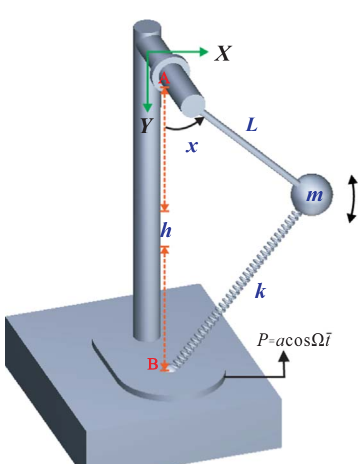

Fig. 1. Physical model of the parametrically excited pendulum with irrational nonlinearity.

$$mLx'' + CLx' + m(g + a\Omega^2\cos\Omega\bar{t})\sin x + kh\sin x\left(1 - \frac{l}{\sqrt{L^2 + h^2 - 2Lh\cos x}}\right) = 0. \tag{4}$$

where prime denotes derivative with respect to $\bar{t}$. x is the angular displacement, m is the mass of the pendulum, g is the acceleration due to gravity, L is the length of the pendulum rod, k and l are the stiffness and relax length of the spring, h is the height from **A** to **B**, C is the viscous damping coefficient and a vertical harmonic oscillation is written as a function of the time $\bar{t}$, $P = a\cos\Omega\bar{t}$. It can be written in a different form if a new non-dimensional time t is introduced as follow

$$\ddot{x} + \gamma\dot{x} + (1 + p\cos\omega t)\sin x - \frac{q\sin x}{\sqrt{1 + \lambda^2 - 2\lambda\cos x}} = 0, \tag{5}$$

where

$$t = \sqrt{\frac{mg + kh}{mL}}\bar{t},\quad \gamma = \sqrt{\frac{mL}{mg + kh}}\frac{C}{m},\quad p = \frac{ma\Omega^2}{mg + kh},$$
$$\omega = \sqrt{\frac{mL}{mg + kh}}\Omega,\quad \lambda = \frac{h}{L},\quad \alpha = \frac{h}{l},\quad \kappa = \frac{kl}{mg},\quad q = \frac{\kappa\lambda}{1 + \kappa\alpha}.$$

From the mathematical perspective, parameters λ, α and κ can be regarded as mutually independent and then a combination q is introduced in system (5). In this case, the corresponding value of q can be calculated by changing independently the length of the pendulum and stiffness of the spring. From the physical point of view, parameter λ defines the geometry of the model and q mainly reflects the stiffness of spring. And most particularly, the classical parametrically excited pendulum can be obtained directly by changing the parameter k to 0 in system (4). It is worth pointing out that this pendulum system exhibits both smooth to discontinuous dynamics depending on the value of the smoothness parameter λ.

3. Unperturbed dynamics

In this section, the unperturbed dynamics are analyzed directly for the original equation with irrational nonlinearity by using nonlinear dynamical technique [22] in both smooth and discontinuous cases.

When p=0 and $\gamma = 0$, the unperturbed system of the novel parametrically excited pendulum (5) is described by

$$\ddot{x} + \sin x - \frac{q\sin x}{\sqrt{1 + \lambda^2 - 2\lambda\cos x}} = 0, \tag{6}$$

which is smooth for $\lambda > 1$ and discontinuous at x=0 for $\lambda = 1$. It is worth reiterating here that the discontinuous dynamics is obtained by changing the parameter λ to 1 smoothly, which is the limit case as $\lambda \to 1$ from the mathematical point of view. Letting $\ddot{x} = F(x)$, the restoring forces of smooth and discontinuous case can be obtained as follows

$$F(x) = \begin{cases} -\sin x + \dfrac{q\sin x}{\sqrt{1 + \lambda^2 - 2\lambda\cos x}}, & \lambda > 1, \\ -\sin x + q\cos\frac{x}{2}\mathrm{sign}\left(\sin\dfrac{x}{2}\right), & \lambda = 1. \end{cases} \tag{7}$$

Even the stiffness of the spring is linear and the resistance force supplied to the system is strongly irrational nonlinearity due to geometry configuration. Letting $F(x) = 0$, the equilibria of system (6) can be obtained as follows

$$(x_1, y_1) = (0, 0),\quad (x_{2,3}, y_{2,3}) = (\pm\pi, 0),$$
$$(x_{4,5}, y_{4,5}) = \left(\pm\arccos\left(\frac{1 + \lambda^2 - q^2}{2\lambda}\right), 0\right), \tag{8}$$

where $x_{4,5}$ exist if and only if $|1 + \lambda^2 - q^2| < 2\lambda$. To understand the influence of parameters λ and q in system (6), we construct the