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Brachistochrone with Coulomb friction as the solution of an isoperimetrical variational problem

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ABSTRACT

The problem of finding the plane supporting curve (brachistochrone) along which a heavy particle released from rest at the given starting point slides with dry (Coulomb) friction and reaches the given destination point in least time, is stated as a variational isoperimetrical problem. The finite parametric equations of the extremal curve are obtained. This curve is the sought-for brachistochrone if the solution of the problem exists. Several numerical examples are given.

1. Transformation of the equations of motion

Let Oxy be the Cartesian right-oriented system of coordinates with the y-axis pointing straight down and the origin O that is the initial extreme of a plane curve along which a heavy particle of mass m slides. We write the equations of motion of the particle provided that, besides of the normal reaction force, the tangent dry friction force acts on the particle from the supporting curve.

Denote by $\mathbf{r}(t) = \{x(t),\ y(t)\}$ the parametric equations of the supporting curve (t is the time). The equations of motion of the particle are

$$m\frac{dv}{dt} = mg\,\mathbf{j}\boldsymbol{\tau} - kN, \qquad \frac{mv^2}{\rho} = mg\,\mathbf{j}\mathbf{n} + N, \tag{1}$$

where N is the modulus of the normal force of pressure acting upon the particle from the curve, the coefficient of friction $0 < k < 1$, the quantity $\rho > 0$ is the radius of curvature of the curve at the current point $P\{x(t),\ y(t)\}$, $\boldsymbol{\tau}$ and $\mathbf{n}$

$$\left\{\frac{\dot{x}(t)}{v},\ \frac{\dot{y}(t)}{v}\right\}, \quad \left\{\frac{\dot{y}(t)}{v},\ \frac{-\dot{x}(t)}{v}\right\} \quad \left(v = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\right)$$

are the unit vectors of the tangent and normal lines at this point, $\mathbf{j}$ is the orth of the downward vertical and g is the acceleration of free fall.

By elimination of N from (1) we obtain the differential equation for the particle motion

$$\frac{dv}{dt} + \frac{kv^2}{\rho} = g\,\mathbf{j}(\boldsymbol{\tau} + k\mathbf{n})$$

which can be rewritten in the form

$$\frac{dv}{dt} - kv\dot{\varphi} = g(\sin\varphi - k\cos\varphi), \tag{2}$$

since

$$-\frac{1}{\rho} = \frac{\dot{\varphi}}{v}, \qquad \varphi = \arctan y'$$

(for definiteness, we assume that the supporting curve is convex down).

The Eq. (2) is linear with respect to the modulus v of the particle velocity but this equation cannot be solved until the function $\varphi(t)$ is unknown. Nevertheless, it is possible to represent v as

$$v(\varphi) = C(\varphi)e^{k\varphi}, \quad C'e^{k\varphi}\dot{\varphi} = g(\sin\varphi - k\cos\varphi). \tag{3}$$

Therefore

$$dt = \frac{e^{k\varphi}dC(\varphi)}{g(\sin\varphi - k\cos\varphi)} \tag{4}$$

where the function $C(\varphi)$ is unknown till the supporting curve will not be given explicitly.

Let us use the expressions (3) and (4) in order, by means of the classical Calculus of variations technique, to solve the problem of finding the curve of least descent for a heavy particle which starts at the point $O(0,\ 0)$ without initial velocity and reaches the given destination point $Q(a,\ b)$ $\quad (a \geq 0,\ \ b > 0)$.

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2. Previous history of the problem

About hundred of papers on the problem of the brachistochrone (frictionless or with dry (Coulomb), viscous and other friction) and its generalizations has been published up to now. It seems the paper [1] is the first publication on the brachistochrone with dry friction after the appearance of I. Bernoulli's statement of the well-known problem on the frictionless brachistochrone and after the appearance of Euler's work on the bruchistochrone with viscous friction.

Almost all these publications can be divided in two groups with respect to methods used to solve the problems. These are methods of Calculus of variations and of the Theory of optimal control.

In [1] the problem was solved as the conditional variational problem of minimization of the functional (the notations are in general usage)

$$\int_{\tau_1}^{\tau_2} t' d\tau$$

under restrictions

$$(x'^2 + y'^2)^{1/2} - vt' = 0 \quad \text{and} \quad vv' + gy' + kgx' + k(x'y'' - y'x'')/t'^2 = 0$$

imposed upon the system. Here the prime denotes a derivative relative to the auxiliary time τ which is chosen so that any infinitesimally close curve of comparison passes through the given points O and Q at the same moments τ_1 and τ_2 correspondingly. The problem has been reduced to the numerical solving one strongly nonlinear equation with respect to the ratio of two parameters. After the substitution of the evaluated parameters in the finite parametric equations of the extremal curve these equations take the completed form. In the example of [1] the curves specified by such a way have been accepted as the required brachistochrones. The general questions on the existence, uniqueness and optimality of the solution were not concerned in [1].

Almost simultaneously with the publication of [1] in the paper [2] the problem on the brachistochrone with dry friction was treated as the problem on the regular perturbated frictionless brachistochrone. The coefficient of friction is the small parameter.

It seems the problem on the brachistochrone with dry friction has been solved first as the problem of optimal control in [3]. Moreover, the generalized statement when the particle starts with nonzero velocity, is considered. In [3] control is introduced as the angular velocity of the slope angle of the tangent to the supporting curve: $\dot{\theta} = \Omega$. Since the control Ω enters the Hamiltonian $H(x, \lambda, \Omega)$ (x and λ are correspondingly the vectors of state and costate variables) in linear way Pontryagin's maximum principle does not work and it is necessary to use methods of the theory of singular optimal control. According to this theory the following set of equations serves to specify the required control

$$\frac{d^i}{dt^i}\left(\frac{\partial H}{\partial \Omega}\right) \equiv 0, \quad \frac{d^m}{dt^m}\left(\frac{\partial H}{\partial \Omega}\right) = g(x, \lambda, \Omega) = 0 \qquad (i = 0, 1, \ldots, m-1)$$

(m is always an even number). The inequality

$$(-1)^{\frac{m}{2}} \frac{\partial}{\partial \Omega}\left[\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial \Omega}\right)\right] \geq 0,$$

called the generalized Legendre-Clebsh or Kelley condition, is the necessary condition of optimum. In general, any sufficient conditions for optimality of the singular control are unknown.

In [3] the finite equations of the extremal curve are obtained and these equations are adopted to the case of the classical statement of the problem with zero initial velocity of the moving particle.

The publication [4] has the elementary mistake, namely, into the equality for variation of the mechanical energy of the particle that slides along the supporting curve with dry friction, the expression for magnitude of the particle velocity has been substituted from Snell's law. But I. Bernoulli used this expression while solving the problem on the *frictionless* brachistochrone.

In Yu. Golubev's paper [5] the problem on the brachistochrone with dry friction is solved with the help of the theory of singular optimal control but in contrast with [3] the control $u(t)$ has been introduced as ratio of the magnitudes of the normal force and of the particle momentum. The control for the extremal curve and the finite parametric equations of the extremal curve are obtained. The right-hand sides of these equations depend on $u(0)$. At the initial point of the brachistochrone the conditions v=0 (according to the problem statement) and N=0 (the tangent to the brachistochrone is vertical) fulfill. Hence the ratio $u(0)$ represents the uncertainty of $\frac{0}{0}$ type. Putting a set of numerical values to the quantity $u(0)$ Yu. Golubev gives the geometrical illustrations of some characteristics for sliding of the particle along the corresponding extremal curves.

In papers [5,7] more general problem on the brachistochrone with forces of dry and viscous friction acting upon the particle simultaneously is also solved.

Jeffrey Hayen [6] succeeded in obtaining, in our opinion, the most simple and completed form of the finite equations of the extremal curve in the classical problem on the brachistochrone with dry friction. He has developed the method of [1] logically for two cases when the independent variable is the time and when the independent variable is the horizontal coordinate of the sliding heavy particle. It should be emphasized the next result of [6] which probably has been proved for the first time: if the final point lies inside of the angle 2γ ($\gamma = \pi/2 - \alpha$, α is the angle of friction) with the vertex at the start point and with the bisectrix directed downwards then the unique extremal curve connected these two points exists. The necessity of this condition was established in [1] but its sufficiency has been reproved several times (see e.g. [7]).

The paper [8] of Slaviča Šalinić is devoted to some generalizations of the classical problem on the brachistochrone with dry friction when the initial velocity of the particle is nonzero. In the first statement the initial and final positions of the particle are given. In the second statement the starting point is given but the destination point lies on a given vertical straight line. Such generalizations has already been considered earlier (see e.g. [3,6]) but in Šalinić's statement of the problem it is also supposed that the constraint is *unilateral*, i.e. the particle can move freely. This condition gives the additional nonlinear nonholonomic constraint that with the equations of the particle motion which are considered as the imposed constraints, generates the necessary conditions for minimum of the descent time. The problem is reduced to the solution of a finite system with a high nonlinearity by means of modern numerical methods.

In [9,10] this statement of the problem is treated with the help of Pontryagin's maximum principle provided that the normal force magnitude belongs to the given segment.

The given synopsis does not concern a lot of papers in which there are considered the problem on the brachistochrone in different force fields (active forces), on motion of the heavy particle between two given points on two-dimensional surfaces with dry friction, on motion of the particle under the drag force (Euler's problem and more general), the relativistic statement of the problem, dynamics of a rigid body or of a chain of coupling rigid bodies having a point (permanent or not) that touches the supporting curve with dry friction, and others.

Below the new way of solving the classical problem on the brachistochrone with dry friction is given. The problem is considered as the variational problem with two isoperimetrical conditions. This approach leads to the observable form of the finite equations for the extremal curves. It is appeared that there are two different extremal curves in the problem. Both of them satisfy necessary Euler's equation and the given boundary conditions. But one of these curves corresponds to the irrelevant solution of the problem. If the solution of the problem exists the sought-for brachistochrone is the another curve.