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The study of equivalent material parameters in a hyperelastic model

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ABSTRACT

We study a hyperelastic model of some biological soft tissues with emphasis on the problem of its matching with the material parameters acquired by experimental mechanical tests. First, we study the polyconvexity property of the hyperelastic model. Then, we explore the notion of equivalent sets of material parameters. We perform a numerical study of the regions of equivalent material parameters characterizing the curves predicted by the hyperelastic model that are close, within a prefixed tolerance, to those given by the experimental data. In the numerical study we use the quadratic variation and the Hausdorff distance. The study suggests that a qualitative knowledge of shape and volume of the regions of equivalent material parameters can provide both a criterion for the optimal match between the model with the experimental data and an indication on the reducibility of the number of parameters used in the model.

1. Introduction

1.1. An overview of soft tissue modeling

Many biological soft tissues can be described by hyperelastic models, viscoelastic models, or even by more general models (see, e.g., [6,9,13,26,28,29] and their references). As a consequence, a lot of experiments can be numerically simulated and the related material parameters implemented in the model describing tissue properties can be adjusted in order to recover the experimental data.

Nevertheless, there are several open issues about inverse approaches in hyperelasticity, as shown in the literature (see, e.g., [1] and references therein). We first point out that an inverse problem is usually related to a model which can be defined by different possible families of material parameters. Hence, once the experimental data and a related model have been fixed, one must adopt an optimal approach in order to localize the material parameters.

The first general domain of parameters must provide physically reasonable material behavior of the model. It is well known that for hyperelastic models such a behavior is guaranteed by the polyconvexity condition of the strain energy function with respect to the deformation gradient (see, e.g., [7,20,25,2,3,26], and their references). The second more specific domain of parameters must match a known family of stress-strain curves given by the experimental data obtained for the specific tissue. With respect to this issue, various approaches can be given. Often a cost function and a suitable algorithm are employed to evaluate and minimize the differences between experimental data and model simulations. For example, the particular simulated annealing algorithm, developed in [8], has been proved suitable for the complex behavior of the cost function. Optimization algorithms must be further enhanced because of the multimodal behavior of the cost function. Thus, coupled deterministic - stochastic algorithms have been developed (see for example [17,18] and the references therein). The approach allows both to minimize the discrepancy between experimental and model results and to explore the domain of admissible parameters.

In another type of approach an easy fitting of polyconvex stored energies can be applied to soft tissues with no optimization procedure. This is shown for example in [3]. We stress that in both approaches the final target is to find at least one vector of constitutive parameters for one given tissue, thus avoiding a complete analysis of the model and its dependence from the parameters.

1.2. On the determination of material parameters

In view of the different determination procedures of parameters outlined above, we stress that another related meaningful issue is that of uniqueness of the constitutive parameters for a given tissue. In fact, a given set of experiments on a soft tissue can provide only a certain number of limited information about the mechanical behavior of a material. The vector of the material parameters might be modified by increasing the set of the known experiments. This leads to the issue of

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the non-uniqueness of the parameters inferred by the model. This is an open problem for many hyperelastic models of tissues, and it seems that there is not a general solution to it (see, e.g. the different approaches in [1,10,12,14,15,22,26,27] and the references therein). We also remark that increasing the number of experimental data could be useful to recover uniqueness of the material parameters, but this feature seems to be connected with the particular choice of the model.

In some papers, the issue of non-uniqueness of the material parameters does not play any role because their goal is to represent the characteristic stress-strain curve for a given restricted set of experiments (see, e.g., Sect. 4.1 in [2]).

For the hyperelastic model presented here, we consider the problems of the determination of the material parameters and also of their uniqueness (see Section 3.1). A fully incompressible hyperelastic model and an almost incompressible hyperelastic model are used to identify the constitutive parameters on different kind of tissues.

This approach can be extended also to other hyperelastic models, viscoelastic models, and also to more general models.

1.3. Outline of the results

In what follows, we provide an overview of the main results of the paper with respect to the above discussed topics.

In Section 3.1 we introduce the hyperelastic model with the strain energy function, related invariants, and constitutive parameters. In Section 3.2 we prove the property of polyconvexity of the strain energy function $W = W(\mathbf{C}, \omega)$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green strain tensor, $\mathbf{F}$ is the deformation gradient and $\omega \in \mathbb{R}^N$ is the vector whose components are the constitutive parameters (here $N = 2, 4, 6$). It turns out that W is twice continuously - differentiable with respect to $\mathbf{C}$ and continuous with respect to ω.

It is well known that polyconvexity of the energy guarantees the physical behavior of the model. In our model such property holds when the set of constitutive parameters $\Omega \subset \mathbb{R}^N$ has the form

$$\Omega = \{(\omega_1, \ldots, \omega_N) = \omega \in \mathbb{R}^N | \omega_i > 0\}. \tag{1}$$

Thus within this domain we look for the regions of parameters that guarantee the match between the model

$$\mathbf{P}(\mathbf{F}, \omega) = 2\mathbf{F}\, \partial_{\mathbf{C}} W(\mathbf{F}, \omega). \tag{2}$$

(first Piola-Kirchhoff tensor) and the experimental data.

As it is known, in the fully incompressible case we have $\mathbf{P} = -p\, \mathbf{F}^{-T} + 2\mathbf{F}\, \partial_{\mathbf{C}} W$ where p is a Lagrange multiplier.

For a given family of deformation gradients $\{\mathbf{F}_j : \ 1 \le j \le M\}$, let y_j be the mean value of the experimental data set caused by the deformation $\mathbf{F}_j$, which is taken on a given specimen. Then we look for a vector ϖ that minimizes the cost function

$$\sigma(\omega, y) := \frac{1}{M} \sum_{j=1}^{M} \left| 2 - \frac{y_j}{P_{11}(\mathbf{F}_j, \omega)} - \frac{P_{11}(\mathbf{F}_j, \omega)}{y_j} \right|^2 \tag{3}$$

within a prefixed tolerance.

This is done by a coupled stochastic-deterministic algorithm of the kind briefly discussed above and used for similar models in, e.g., [7,20] and in the references therein. In particular, a (stochastic) simulated annealing algorithm to find a first approximated local minimum. Then, we apply a (deterministic) simplex method to find a local minimum.

After that, we need to study all the possible other material parameters $\widetilde{\omega}$ which are, in a fixed bounded region of Ω, equivalent to ϖ. We first select $\widetilde{\omega}$ in such a way that

$$\sigma(\widetilde{\omega}, y) \le \sigma(\varpi, y). \tag{4}$$

In addition, as a second requirement we make use of the so-called Hausdorff distance d_H between sets (see for example [4] and the references therein). Let us recall that the Hausdorff distance between two sets $X, Y \subset \mathbb{R}^2$ with Euclidean distance on $\mathbb{R}^2$, is defined by

$$\mathbf{d}_{\mathrm{H}}(X, Y) := \max\{\mathbf{d}(X, Y); \mathbf{d}(Y, X)\}\, \mathbf{d}(X, Y) := \sup_{x\in X} \inf_{y\in Y} d(x, y). \tag{5}$$

It is used to compare the graphs of the curves relating the experimental data to the model. It guarantees a satisfactory analysis of the tissues. Now, let $\mathcal{G}(\omega) \subset \mathbb{R}^2$ be the graph of the curve given by $P_{11}(\mathbf{F}, \omega)$ for $\mathbf{F} = \mathbf{F}_j$ and fixed ω, and let $\mathcal{Y} \subset \mathbb{R}^2$ be the graph of the experimental curve with values y_j; then the inequality

$$\mathbf{d}_{\mathrm{H}}(\mathcal{G}(\widetilde{\omega}), \mathcal{Y}) \le \mathbf{d}_{\mathrm{H}}(\mathcal{G}(\varpi), \mathcal{Y}) \tag{6}$$

determines the second selection of the parameters $\widetilde{\omega}$. As we will show in Section 3.3, various two dimensional regions of parameters can be displayed by looking at different material parameters satisfying (4) and (6). As explained above, the parameters $\widetilde{\omega}$ are minimizers of the cost function σ as well as the parameter ϖ. However, in order to determine the set of parameters $\widetilde{\omega}$ we need first to determine at least a parameter ϖ and then we can write inequalities (4)–(6) and call a related routine that determines numerically the set of $\widetilde{\omega}$. This numerical study is obtained through MATLAB® (software house MathWorks). The related algorithm can be easily adjusted for different stress tensors and thus for different hyperelastic models. A qualitative analysis of such regions of parameters, for different experimental data, provides useful information about the mechanical properties of the tissue under study and also some indicators of the optimality of the chosen hyperelastic model.

The novelty of our study is to show the possibility of a more complete analysis of the material parameters of a soft tissue (here in particular urethral tissue). This involves not only the determination of a (vector of) material parameter ϖ as done previously, but in addition the determination of the larger set of parameters providing the same minimization of cost function involving the model and the experiment.

2. Settings and preliminaries

In this section we provide a summary of notations used in the paper together with some central definitions of hyperelasticity.

We denote by Lin$^+$ the set of all second-order tensors with positive determinant, to be identified with the family of 3×3 matrices $\mathbb{M}_{n\times n}$ with positive determinant. The set Orth$^+$ is the subset of Lin$^+$ given by rigid rotation tensors $\mathbf{R}$, namely such that $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, where $\mathbf{I}$ is the unit tensor. The set Sym$^+$ is the subset of Lin$^+$ of symmetric tensors $\mathbf{U}$, i.e. such that $\mathbf{U} = \mathbf{U}^T$.

Following the notations of continuum mechanics, here $\mathbf{F} \in$ Lin$^+$ denotes the deformation gradient, $\mathbf{C} = \mathbf{F}^T\mathbf{F} \in$ Sym$^+$ denotes the right Cauchy-Green deformation tensor, the map $\mathbf{F} \in \mathrm{Lin}^+ \longmapsto W(\mathbf{F}^T\mathbf{F}) \in \mathbb{R}$ is the strain energy function, and the first Piola-Kirchhoff stress tensor reads $\mathbf{P} = 2\mathbf{F}\partial_{\mathbf{C}} W$. We refer the reader to the standard textbooks of continuum mechanics [13,23,24,28].

Definition 2.1. A map $\mathbf{F} \in \mathrm{Lin}^+ \longmapsto W(\mathbf{F}) \in \mathbb{R}$ is said to be convex if

$$W(\tau\mathbf{F}_1 + (1-\tau)\mathbf{F}_2) \le \tau W(\mathbf{F}_1) + (1-\tau) W(\mathbf{F}_2) \tag{7}$$

for every $\mathbf{F}_1$, $\mathbf{F}_2$ and τ with $0 \le \tau \le 1$.

In the following, we recall some generalized convexity conditions. In particular, from [23] we recall

Definition 2.2. A map $\mathbf{F} \in \mathrm{Lin}^+ \longmapsto W(\mathbf{F}) \in \mathbb{R}$ is said to be polyconvex if there exists a function P: $\mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ such that

$$W(\mathbf{F}) = P(\mathbf{F}, \mathrm{Adj}[\mathbf{F}], \det[\mathbf{F}]) \tag{8}$$

and $(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) \in \mathbb{R}^{19} \longmapsto P(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) \in \mathbb{R}$ is convex.

The policonvexity property is usually used to select physically reasonable models (see, e.g., [26] and the references therein).

Definition 2.3. A twice differentiable function $\mathbf{F} \in \mathrm{Lin}^+ \longmapsto W(\mathbf{F}) \in \mathbb{R}$ fulfills the Legendre-Hadamard condition if $\forall\, a, b \in \mathbb{R}^3$, $\forall\, \mathbf{F} \in$ Lin$^+$ we have

$$\mathrm{D}^2_{\mathbf{F}} W(\mathbf{F}).\, (a \otimes b, a \otimes b) \ge 0. \tag{9}$$