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A hybrid numerical method and its application to inviscid compressible flow problems

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ABSTRACT

An improved version of the artificially upstream flux vector scheme, is developed to efficiently compute inviscid compressible flow problems. This numerical scheme, named AUFSR (Tchuen et al. 2011), is obtained by hybridizing the AUFS scheme with Roe's solver. This approach handles difficulties encountered by the AUFS scheme, in the case where the flux vector does not check the homogeneous property. The present scheme for multi-dimensional flows introduces a certain amount of numerical dissipation to shear waves, as Roe's splitting. The AUFSR scheme is not only robust for shock-capturing, but also accurate for resolving shear layers. Numerical results for 1D Riemann problems and several 2D problems are investigated to show the capability of the method to accurately compute inviscid compressible flow when compared to AUFS, and Roe solvers.

1. Introduction

In a progressive effort to solve complex flow problems described by Euler/Navier–Stokes equations, the research on how to maximize both accuracy and efficiency has been the primary goal of designing an algorithm in numerical analysis. The compressible flow problems involve complex flow phenomena, such as strong shock waves, shock–shock interactions and shear layers. A number of numerical flux functions for inviscid fluxes have been devised as approximate solutions to the Riemann problem. Among these formulations, upwind numerical methods have become popular for solving hyperbolic partial differential equations with discontinuous solutions. They take into account the physical properties of the flows into the numerical formulation, and their essential characteristics are their particular treatment of the convective terms into Navier–Stokes equations from a well adapted flux decomposition. These are usually classified as either flux difference splitting (FDS) or flux vector splitting (FVS), each method having its advantages and disadvantages.

The FDS scheme is based on the difference between the decomposition of fluxes, constructed on an approximated solution of the local Riemann problem between two adjacent states. Several formulations of FDS schemes have been compared in the literature [1–5]. These methods have been proven to be capable of capturing sharply and correctly the discontinuities of all types such as shock waves corresponding to linear as well as non-linear waves. Unfortunately, certain FDS schemes contain subtle flaws and produce spurious solutions like low frequency post shock fluctuations in the case of slowly moving shock, carbuncle phenomena, odd–even decoupling and kinked Mach stem on certain occasions [2,6–10]. Even though the FVS methods [11–13] when modified to improve their capability to capture contact discontinuities are not spared from shock instabilities and carbuncle phenomena [9,14], they rely on a decomposition of the flux vector into the positive and negative components and according to the sign of the propagation of the associated waves. They are found to be very successful in capturing steady discontinuities represented by nonlinear waves, which include shocks. However, they are not effective in capturing the discontinuities represented by linear waves, which result in incorrect diffusion of the contact surface and shear waves and a high dissipation in strong rotational flows [9,15]. However, when compared with Godunov-type schemes, the FVS results are poorer resolutions of discontinuity, particularly contact discontinuity [13]. Most upwind schemes, either Godunov-type or FVS methods, have difficulties in resolving the sonic point, and produce a spurious expansion shock there.

Roe's [2] approach, which is an FDS scheme, is widely used because of its accuracy, quality and mathematical clarity. However,

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the scheme may sometimes lead to unphysical flow solutions in certain problems. They admit rarefaction shocks that do not satisfy the entropy condition. This flaw can be easily handled by simple entropy fix procedures in a one-dimensional case. To improve the accuracy of solution of these problems, Quirk [8] also pointed out that Roe's original scheme should be modified or replaced by other schemes in the vicinity of a strong shock. The combination of Roe's scheme with other solvers, would therefore be promising.

The AUFS method, which is a special FVS scheme, was proposed by Sun and Takayama [13] for splitting flux vectors of Euler equations. This scheme has recently been extended [16] to calculate two-dimensional hypersonic flow, in thermochemical nonequilibrium. The AUFS scheme introduces two artificial wave speeds into the flux decomposition. The direction of wave propagation is adjusted by these two wave speeds. One part of the flux is numerically obtained following the Steger–Warming approach, which is an FVS method. Steger and Warming [11] were the first to use the homogeneous property of the governing equations of gas, and expressed the inviscid flux vectors in terms of their Jacobian matrices. This method is simple, accurate and robust for the Euler equations but it is not applicable to non-homogeneous equations (e.g., the magnetohydrodynamics (MHD) equations).

Hence, the current approach is motivated by the desire to combine the efficiency of FVS schemes and the accuracy of FDS schemes. In this paper, the flux splitting scheme named the AUFSR scheme [17], is obtained by hybridizing the AUFS solver (FVS) and Roe's scheme (FDS). The motivation in using Roe's scheme relies on several advantages, which make it well known. The resulting flux function can be implemented in a very simple manner, in the form of Roe's solver with modified wave speeds, so that converting an existing AUFS flux into the new fluxes is an extremely simple task. The procedure is to avoid difficulties encountered by the AUFS scheme, when the artificial diffusion is computed with the Steger–Warming approach in the case where the flux vector does not check the homogeneous property. The AUFSR scheme should be able to solve flows with non-homogeneous properties, furthermore, this type of hybrid method can be easily extended to a higher order of accuracy through the formal use of MUSCL interpolations.

In this paper, a hybridized method to construct a simple, accurate and robust scheme is presented and used for solving Euler equations. Numerical examples are given to demonstrate that the hybrid scheme has a high computational accuracy when compared to AUFS, Roe's and exact Riemann solvers.

2. Preliminaries

Consider as an initial-value problem, the one-dimensional (1D) system of conservation laws for ideal-gas flows,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad t > 0, -\infty \leq x \leq \infty \tag{1}$$
$$U(x, 0) = U_o(x)$$

where U and F are the vectors of conserved variables and fluxes, given respectively by, $U = (\rho, \rho u, E)^T$ and $F = (\rho u, \rho u^2 + p, (E + p)u)^T$. Here ρ is density, p is pressure, u is particle velocity and E is the total energy per unit volume defined as, $E = \rho(e + u^2/2)$. The ideal-gas equation of state is assumed, $p = (\gamma - 1)\rho e$, where $\gamma = 1.4$. The flux vector can be rewritten as

$$F = uU + P \tag{2}$$

where $P = (0, p, pu)^T$. For the numerical solution of (1), we shall consider piecewise a constant approximation of U_i^{n+1} defined by the explicit three-point scheme in conservative form,

$$U_i^{n+1} = U_i^n - \lambda(F_{i+1/2} - F_{i-1/2}) \quad n \in N, i \in Z \tag{3}$$

where $\lambda = \Delta t/\Delta x$, Δt is the time step, and Δx is the grid size. $F_{i+1/2}$ is the numerical flux vector defined by two neighboring cells (or to the left or right cell),

$$F(U_{i+1/2}) = F(U_i, U_{i+1}) = F(U^L, U^R). \tag{4}$$

In the finite-volume formulation, the difference among all the numerical schemes lies essentially in the definition of the numerical flux $F_{i+1/2}$ evaluated at the cell interface. The numerical flux will vary, depending on the properties it must meet. The system of Eqs. (1) can be expressed in quasi-linear form

$$U_t + AU_x = 0 \tag{5}$$

where

$$A = \frac{\partial F}{\partial U} \tag{6}$$

is the Jacobian matrix. If A has a complete set of linearly independent eigenvectors, the eigenvalues associated with one subvector can be all positive and those associated with the other all negative. The system (1) is homogeneous because it satisfies $F(U) = A(U)U$.

3. General form of the AUFS solver

The fundamental idea of the AUFS scheme, is to split the flux vector (2) as follows [13]:

$$F = (1 - M)[(u - s_1)U + P] + M[(u - s_2)U + P] \tag{7}$$

where s_1 and s_2 are two scalar constants. $M = s_1/(s_1 - s_2)$, and (7) can be rewritten as

$$F = (1 - M)F_1 + MF_2 \tag{8}$$

with $F_{1,2} = (u - s_{1,2})U + P$. These two flux vectors are different from the original by a term $-sU$. Their Jacobian matrices are $A_{1,2} = \partial F_{1,2}/\partial U = A - sI$ and their corresponding matrices of eigenvalues become $\Lambda_{1,2,3}$ = diagonal$(u - s - c, u - s, u - s + c)$. An excellent merit of the AUFS scheme is that the eigenvalues of $\lambda_{1,2,3}$ can be changed by varying the scalar value s. s is an artificially introduced wave speed. Depending upon the value and sign of s_1 that one gets

$$F_1 = \frac{1}{2}(P^L + P^R) + \delta U \quad \text{and} \quad F_2 = U^\alpha(u^\alpha - s_2) + P^\alpha. \tag{9}$$

The performance of the scheme should rely on the way to express δU which represents the artificial viscosity. In the AUFS scheme, this viscosity is determined by the Steger–Warming formula. Substituting (8) and (9) to (7), the following final intercell flux is obtained

$$F = (1 - M)\left[\frac{1}{2}(P^L + P^R) + \delta U\right] + M\left[U^\alpha(u^\alpha - s_2) + P^\alpha\right] \tag{10}$$

where

$$\alpha = \begin{cases} L & \text{for } s_1 > 0, \\ R & \text{for } s_1 \leq 0. \end{cases} \tag{11}$$

The artificial numerical velocities s_1 and s_2 make it possible to write the intercell flux. The expressions proposed by Sun and Takayama [13] are given as follows.

$$s_1 = \frac{1}{2}(u_L + u_R) \tag{12}$$

and

$$s_2 = \begin{cases} \min(0, u_L - c_L, u^* - c^*) & \text{for } s_1 > 0 \\ \max(0, u^* + c^*, u_R + c_R) & \text{for } s_1 \leq 0 \end{cases} \tag{13}$$

with the speed u^* and the sound speed c^* given by [13]

$$\begin{aligned} u^* &= \frac{1}{2}(u_L + u_R) + \frac{c_L - c_R}{\gamma - 1}; \\ c^* &= \frac{1}{2}(c_L + c_R) + \frac{1}{4}(\gamma - 1)(u_L - u_R). \end{aligned} \tag{14}$$