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The viscous torsional pendulum

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ABSTRACT

A disk that is free to rotate about its axis and connected to a torsional spring behaves as a damped oscillator when twisted and released. The initial elastic energy is periodically turned to kinetic energy and it gets progressively dissipated by the viscous friction exerted by the surrounding fluid. The subsequent oscillating motion is dictated by the fluid–solid interaction which is here solved numerically by coupling the second Euler's law, that prescribes the disk's rotation, to the Navier–Stokes equations, that govern the fluid's motion. Two different regimes are observed: (i) a low-amplitude regime, where the phase lag between the twisting velocity and the viscous torque is equal to $3\pi/4$ and the instantaneous damping rate is constant; (ii) a higher amplitude regime, where the twisting velocity and the viscous torque are out of phase and the damping rate increases proportionally to the square root of the oscillation amplitude.

These observations are rationalized through boundary layer theory applied in the vicinity of the disk, thus retrieving analytical expressions of the viscous torque available in the literature. By using a multiple scale technique, an explicit expression for the free decay of the disk torsional pendulum is obtained which well predicts the results of the numerical simulations without any tunable parameter.

1. Introduction

A torsional pendulum consists of a disk-like mass suspended from a thin rod or wire. When the mass is twisted around the axis of the wire, the wire exerts a torque on the mass, tending to rotate it back to its original position. If twisted and released, the mass will oscillate back and forth periodically, and the system's natural frequency can be controlled by accurately tuning the oscillating mass and the restoring force exerted by the wire.

Due to its simple and regular dynamics, the torsional pendulum has been used in numerous precision experiments in electrical science, biophysics, petrology, metallurgy, and various other fields of endeavor (Gillies and Ritter, 1993). In particular, it is a mainstay instrument in gravitational physics because it enables one to isolate and measure weak forces with a magnitude comparable with the background gravitational field of the earth (Gundlach and Merkowitz, 2000; Tan et al., 2015). Indeed, if well balanced, it is possible to place the Earth's gravitational force in an orthogonal relationship to the plane in which the signal of interest occurs. The basic idea in these experimental methods is to study gravitational effects by measuring the influence of masses external to the pendulum on its regular periodic motion with operating frequency in the range of 10 μHz–10 mHz.

Torsional pendula are also used in the watch industry as fundamental timing elements. For instance the balance wheel that is the timekeeping device used in mechanical watches, is a peculiar torsional pendulum typically oscillating at 4 Hz where the restoring force is provided by a coiled spring rather than a long twisted wire. The balance wheel and

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hairspring together form a harmonic oscillator, which, due to resonance, oscillates preferentially at a certain frequency. The watchmakers have made constant progress to maintain linearity of the spring even at large amplitude to ensure both a regular time-beating and a sufficient period of operation (Vermot et al., 2011).

In each of these settings, the motion of the torsional pendulum needs to be as regular as conceivable. The presence of any damping source in the system can indeed bias and corrupt the measurements or alter the perfect beating of a sophisticated mechanical watch. Therefore, depending on the application, the planar motion of the disk is ensured by the stiffness of the joints and the disk's rigidity. In addition, mechanical frictions are conveniently suppressed or reduced by using linear torsional springs, lubrication or bearings. However, once the mechanical dissipation sources are dropped, another dissipation mechanism becomes dominant: the viscous friction exerted by the surrounding fluid on the oscillating mass. This dissipative mechanism originates in the relative motion of the solid and the fluid that sets up viscous stresses, which tend to drag the fluid to move with the disk and, as a consequence, prevent the disk's motion. As a result, the resulting damped motion of the disk becomes non-trivial because it depends on the complex motion of the fluid surrounding the disk.

The flow induced by an oscillating disk has been studied extensively since Stokes (1851), who derived an asymptotic solution of the flow field and the torque at the disk's surface that is valid in the limit of high frequency and small amplitude of oscillation, so that the nonlinear effects are neglected. The nonlinear correction to the oscillatory problem was first tackled by Rosenblat (1959) who defined an unsteady inner shear layer at the disk's surface and an outer steady flow region. He reduced the Navier–Stokes equations to a set of one-dimensional ordinary differential equations which correspond to the unsteady version of the classical rotating disk equations solved by Von Karman (1921). By expanding the velocity and pressure in powers of the amplitude, Rosenblat (1959) found an asymptotic solution that approximates the azimuthal velocity and, consequently, the viscous torque at leading order. The same problem has been then studied by Benney (1964), using a multiple scales technique valid over the entire flow domain, while Riley (1965) carried out both the low and the high frequency case by means of a matched asymptotic expansion.

Interested in its fundamental implications, we investigate here the motion of a torsional pendulum consisting of an oscillating disk subjected to viscous friction. The full fluid–structure problem is solved through numerical simulations by coupling the disk's motion to the velocity and pressure field of the fluid, governed by the Navier–Stokes equations. In contrast to classical linear damping theory, we uncover the existence of two different limiting behaviors in dynamics of the system yielding different scalings for the damping rate. This observation is rationalized theoretically in the framework of boundary layer theory. The gained understanding motivates us to introduce a simple phenomenological model for the viscous relaxation of the torsional pendulum. This predictive model gives an analytic expression for the free decay of the system that is compared with the results of the full numerical simulations.

2. Problem description and governing equations

Let us consider a plane disk of radius R, height H and density ρ_d, that is surrounded by a fluid of viscosity μ and density ρ. The disk can only rotate about its axis and is connected to a torsional spring exerting a restoring torque, τ_k, on the disk. The spring is assumed to be ideal, meaning that the torque, τ_k, is proportional to the twisting angle, θ, through the elastic coefficient k,

$$\tau_k = -k\theta. \tag{1}$$

Eq. (1) is the angular version of the Hooke's law. Thus, the disk's position in time is described by a single Lagrangian coordinate, $\theta(t)$, that is the angle of twist of the disk from its equilibrium position. When the disk is twisted by an angle θ_m and then released, the spring exerts a torque on the mass tending to rotate it back to its equilibrium position. As a consequence, the disk oscillates back and forth in the plane perpendicular to its axis, and the potential energy initially stored in the torsional spring is periodically converted into kinetic energy until dissipation brings the system at rest in the equilibrium position, $\theta = 0$. Thus, the motion of the torsion pendulum is described by Euler's second law

$$I\ddot{\theta} + k\theta = \tau_f, \tag{2}$$

where the double dot symbol designates double derivation with respect to time and τ_f is the resultant torque exerted by external forces on the disk. The moment of inertia I depends on the mass geometry and in the case of a disk it is equal to

$$I = \frac{\pi}{2}\frac{H}{R}\rho_d R^5. \tag{3}$$

Eq. (2) is conveniently made nondimensional by using the inverse of the natural frequency of the system, $\omega = \sqrt{k/I}$, and the disk's radius, R, as characteristic time and length scales

$$\ddot{\theta} + \theta = \epsilon\tau_f, \tag{4}$$

where from now on all symbols refer to nondimensional quantities. The number ϵ in Eq. (4) depends on the aspect ratio of the disk, $\frac{H}{R}$, and on the density ratio, $\frac{\rho}{\rho_d}$, according to

$$\epsilon = \frac{2}{\pi}\frac{R}{H}\frac{\rho}{\rho_d}. \tag{5}$$