ARTICLE IN PRESS

Journal of the Mechanics and Physics of Solids ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Stability boundaries for wrinkling in highly stretched elastic sheets

Qingdu Li [a], Timothy J. Healey [b,*]

[a] Key Laboratory of Industrial Internet of Things & Networked Control, Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
[b] Department of Mathematics, Cornell University, Ithaca, NY, USA

ARTICLE INFO

Article history:
Received 30 June 2015
Received in revised form
23 November 2015
Accepted 7 December 2015

Keywords:
Wrinkling
Thin films
Nonlinearly elastic plates
Global bifurcation

ABSTRACT

We determine stability boundaries for the wrinkling of highly unidirectionally stretched, finely thin, rectangular elastic sheets. For a given fine thickness and length, a *stability boundary* here is a curve in the parameter plane, aspect ratio vs. the macroscopic strain; the values on one side of the boundary are associated with stable unwrinkled (flat) states, while stable wrinkled configurations correspond to all values on the other. In our recent work we demonstrated the importance of finite elasticity in the membrane part of such a model in order to capture the correct phenomena. Here we present and compare results for four distinct models: (i) the popular Föppl–von Kármán plate model (FvK), (ii) a correction of the latter, used in our earlier work, in which the approximate 2D Föppl strain tensor is replaced by the exact Green strain tensor, (iii) and (iv): effective 2D finite-elasticity membrane models based on 3D incompressible neo-Hookean and Mooney–Rivlin materials, respectively. In particular, (iii) and (iv) are superior models for elastomers. The 2D nonlinear, hyperelastic models (ii)–(iv) all incorporate the same quadratic bending energy used in FvK. Our results illuminate serious shortcomings of the latter in this problem, while also pointing to inaccuracies of model (ii) – in spite of yielding the correct qualitative phenomena in our earlier work. In each of these, the shortcoming is a due to a deficiency of the membrane part of the model.

1. Introduction

We return in this work to our study of the wrinkling of highly stretched, finely thin, rectangular elastic sheets (cf. Healey et al., 2013). The problem concerns the development of transverse wrinkles as the two opposing shorter, clamped ends are pulled apart. The phenomenon is well known and has been studied from many different points of view (e.g., Friedl et al., 2000; Cerda et al., 2002; Jacques and Potier-Ferry, 2005; Puntel et al., 2011). We refer to the introduction to our earlier paper for an overview of some features of those works. In particular, none of these employs a geometrically exact membrane model. We postpone a discussion of the works of Taylor et al. (2014) and Nayyar et al. (2011, 2014), which employ models closely related to ours, until the end of this section.

Two novel ingredients of our paper from 2013 are: (i) a geometrically exact, elastic membrane model perturbed by very small bending stiffness; (ii) the systematic use of global bifurcation/continuation methods via multiple parameters—

---

* Corresponding author.
E-mail address: tjh10@cornell.edu (T.J. Healey).

http://dx.doi.org/10.1016/j.jmps.2015.12.001

Please cite this article as: Li, Q., Healey, T.J., Stability boundaries for wrinkling in highly stretched elastic sheets. J. Mech. Phys. Solids (2015), http://dx.doi.org/10.1016/j.jmps.2015.12.001

including the macroscopic strain, membrane thickness, and aspect ratio—to uncover stable wrinkling behavior. For a given fine thickness, two key findings are: (1) wrinkling occurs only for aspect ratios contained in a bounded interval, i.e., the stretched membrane remains planar without wrinkles if the aspect ratio is either too large or too small. (2) When wrinkling occurs, as the macroscopic strain is continuously increased, wrinkles first initiate (bifurcate) at a non-zero value, reach a very small, maximum amplitude, diminish, and then disappear altogether. Mathematically this corresponds to an *isola-center* bifurcation diagram (the nontrivial solution curve "starts" and "stops" at two distinct points on the trivial line) (cf. Golubitsky and Schaeffer, 1985). In contrast, the popular Föppl–von Kármán (FvK) model, also considered in Healey et al. (2013), yields *pitchfork* bifurcation diagrams (like the post-buckling curve of an Euler column) for seemingly all aspect ratios – large and small. Observe that a "pitchfork" here unrealistically implies an ever-increasing wrinkling amplitude as the macroscopic stain is increased.

The model employed in Healey et al. (2013) is a correction to the FvK model – Green's strain tensor is employed in lieu of the Föppl approximation. In particular, this results in a geometrically exact membrane model characterized by the Saint-Venant Kirchhoff (S-VK) constitutive law, the mathematical tenability of which, in a highly unidirectional tensile state, is analyzed in our paper. Although the model captures the correct qualitative phenomena, the overall deficiency of that constitutive law as a realistic model for elastomers is clear. One goal of this work is to extend our previous approach to superior constitutive models for membrane elasticity. In particular, we employ 2D models based on the 3D incompressible neo-Hookean (NH) and Mooney–Rivlin (MR) materials (cf. Mooney, 1940; Müller and Strehlow, 2004). Our modeling philosophy here is the same as before, viz, the elastic sheet is idealized as a 2D nonlinearly elastic membrane perturbed by extremely small, linear bending stiffness.

Another motivation for our study comes from the tacit suggestion in Healey et al. (2013) (echoed above) that the FvK model predicts eventual wrinkling – for sufficiently large macroscopic strain – for *all* aspect ratios. In fact we demonstrate here that this is not entirely true, although something along those lines indeed holds, as discussed below. In any case, to efficiently address such questions, our investigation here is focused on the computation of *stability boundaries*: For a given fine thickness and length, the latter is defined as a curve of bifurcation points (engendering wrinkling) in the parameter plane spanned by the aspect ratio and the macroscopic strain. The boundary separates parameter values associated with stable, unwrinkled (flat) states from those corresponding to stable wrinkled configurations. As a basis for comparison, we compute the stability boundaries (for two distinct fine thicknesses) employing the four models mentioned above: FvK, S-VK, NH and MR.

The outline of the work is as follows. In Section 2 we formulate the problem. In addition to the FvK model and the S-VK model, both studied in Healey et al. (2013), we present the two more accurate models, NH and MR. Following the derivation in Müller and Strehlow (2004), for example, the thin membrane is treated as an incompressible, 3D elastomer. The constraint of incompressibility eliminates the through-thickness stretch, while the constitutively indeterminate pressure is employed to achieve zero tractions on the lateral surfaces of the membrane. All of this results in an effective 2D model. Here we tune the parameter(s) so that the shear moduli of the two models agree with that inherent in FvK and S-VK. The same small quadratic bending energy is consistently employed in each of the four models – tuned to the incompressible limit corresponding to Poisson's ratio equal to 0.5. Our justification for this simple choice is the following. All previous studies, both numerical (e.g., Nayyar et al., 2011; Healey et al., 2013; Taylor et al., 2014), and experimental (e.g., Zheng, 2009; Fehér and Sipos, 2014; Nayyar et al., 2014), indicate that the maximum wrinkling displacement is of the same order of magnitude as the fine thickness, while the wavelength of the typical wrinkle is two orders of magnitude greater. We comment further on this in the final section of the paper.

In Section 3 we discuss our strategy for the efficient computation of stability boundaries. Due to the reflection symmetry of the problem across the flat configuration, we demonstrate that the tangent operator evaluated at a planar configuration is block-diagonal, comprising two blocks – one corresponding to the out-of-plane scalar displacement field, the other to the in-plane vector displacement field. In this way we can compute a stability boundary via continuation for an inflated system involving the *reduced* equilibrium equations governing planar configurations combined with the zero-eigenvalue problem for the above-mentioned block associated with out-of-plane displacements. From a general point of view, such a system is known to be convergent by Newton's method (cf. Werner and Spence, 1984). Instead of the latter, however, we solve the two components of the inflated system separately and consecutively. In particular, we determine the zero-eigenvalue component via a simple mid-point-rule iteration, each iterate requiring the updated solution of the reduced equilibrium equations. This avoids the evaluation of second-derivative operators while enabling efficient computation of stability boundaries. The algorithm is summarized in Section 3.

In Section 4 we present our results for two distinct fine thicknesses. For each thickness we first present stability boundaries for the four models – FvK, S-VK, NH, MR – in terms of aspect ratio vs. macroscopic strain. The curves obtained using the FvK model open to the right, predicting a convex and apparently unbounded region of wrinkling. In contrast, each of the three finite-elasticity membrane models, S-VK, NH and MR, yield closed-curve stability boundaries, with the wrinkling regions inside the curves. For the same thickness, the NH and MR boundaries compare very well; the S-VK curves, although closed and thus qualitatively correct, do not compare well with the two more accurate models. We also provide numerous bifurcation diagrams for the various models, featuring the FvK pitchforks and the isloa-centers coming from the other three models. Finally we illustrate some wrinkled configurations along specific solution diagrams for the two new models, NH, MR, considered in this work. As in Healey et al. (2013), we avoid the numerical pitfalls of a near-zero bending stiffness by the simple step of dividing the equations through by the highest power of the thickness and then employing the

Please cite this article as: Li, Q., Healey, T.J., Stability boundaries for wrinkling in highly stretched elastic sheets. J. Mech. Phys. Solids (2015), http://dx.doi.org/10.1016/j.jmps.2015.12.001