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Research paper

# Homogenization of layered media based on scattering response and field integration

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ABSTRACT

The stress wave scattering off of any finite slab may be used as a physical approach to determining some or all of the overall dynamic constitutive parameters of such a specimen. In this paper, this approach is applied to normal incidence of symmetric and asymmetric layered structures with elastic or viscoelastic layers. The method determines all of the overall constitutive constants of a Willis-type material (including zero and non-zero coupling parameters) for all frequencies, even in stop bands of elastic structures, exactly. The ambiguity in phase velocity calculation is overcome using continuity considerations. Symmetries in coupling constants and the restrictions due to energy conservation and dissipation are presented. Integrating the wave equations directly leads to a proposed micro-structural scheme for determination of the overall constitutive parameters. The results of this method are identical to those derived based on the scattering response for all frequencies, including stop bands, and beyond the long wavelength limit of traditional homogenization techniques. The proposed approach has the potential to be applied to 3D-structured unit cells, oblique incidences, and simultaneous scattering of multiple waves.



## 1. Introduction

The primary focus of this paper is to derive accurate and preferably exact overall constitutive descriptions of wave propagation in periodic micro-structured media. This involves the derivation and validation of overall dynamic tensors for which the scattering and flux of energy match those of the macro-scale mechanical waves in a micro-structured medium. Satisfying Hill–Mandel (HM) criterion, i.e. the equivalence of the total micro-scale and overall energy quantities, has been considered a requirement for elastic homogenization approaches (Hill, 1965; Budiansky, 1965). The magnitude of deviation from HM (e.g. due to nonlinear material response) is generally used as the quantity that determines the limit of applicability of homogenization. In the case of dynamic homogenization for stress waves, satisfying HM criterion requires further attention due to the kinetic energy. Some researchers have introduced source terms (e.g. Wang and Sun, 2002) to deal with HM requirement. Even methods including higher order homogenization schemes, such as the non-local approach (e.g. in Hui and Oskay, 2013) will still need to address this issue as discussed in Fafalis and Fish (2015).

The kinetic energy coupling is not the sole complicating factor in dynamic homogenization. While a direct replacement of complex moduli of viscoelastic constituents for elastic ones, as suggested in Hashin (1970), has been quite satisfactory in predicting the overall storage and loss moduli of composites (see for example Nantasetphong et al., 2016a,b), it is not an exact treatment even in the quasi-static limit. The loss of energy will be amplified for higher frequencies due to local deformations and scattering (Qiao et al., 2016). On the other hand, dynamic homogenization approaches (particularly in frequency domain) have the potential to give complex valued overall constants for propagating waves even with fully elastic constituents (Sabina and Willis, 1988). Passivity, a weaker necessary condition than conservation of energy, appears to be violated by some retrieval methods for overall dynamic properties of heterogeneous media as discussed in Simovski (2009) and Srivastava (2015), particularly as the constitutive tensors are treated independently.

The interest in numerical modeling and experimental demonstration of wave propagation and stop bands in the acoustic response of locally resonant 3D periodic media was renewed in the early 2000s; see for example Liu et al. (2000a,b). For this particular case, the authors later revisited the overall mass density estimates based on multiple scattering theory and commented on the need to properly homogenize overall density in addition to the elastic moduli tensor (Sheng et al., 2007). The need to introduce coupling between momentum and strain and between stress and particle velocity, i.e. Willis-type constitutive law, was implied around the same time in Willis (1981a,b) but came into greater

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focus much more recently (Milton and Willis, 2007). Both these observations, i.e. the shortcomings of independent homogenization of the constitutive tensors and the need to introduce coupling ones were observed and discussed earlier for electromagnetic metamaterials and composites. Overall magnetic response was discussed in Pendry et al. (1999), while the first demonstration of negative refraction was given in Shelby et al. (2001). Examples of homogenization of media with chiral matrix and geometry may be found in Wiegelhofer et al. (1997) and Amirkhizi and Nemat-Nasser (2008b), respectively. Calculation of overall properties based on some sort of field averaging has been discussed in Smith and Pendry (2006), Silveirinha (2007), Amirkhizi and Nemat-Nasser (2008a,b) and Andryieuski et al. (2012), while extraction based on scattering response has been discussed in Smith et al. (2002), Chen et al. (2004), Bayatpur et al. (2012) and Cohen and Shavit (2015). Some of these researchers also dealt with bianisotropy, which constitutes coupling between electric and magnetic fields due to the mirror asymmetry of the unit cell (or constituent materials). A number of these references include both approaches as has been done also critically in Simovski and Tretyakov (2007) and Arslanagi et al. (2013).

The derivation of overall constitutive tensors for stress waves in heterogeneous media has been traditionally focused on field averaging techniques (Willis, 2009; Nemat-Nasser et al., 2011; Willis, 2011; Srivastava and Nemat-Nasser, 2012; Norris et al., 2012). Comparisons were made with expected scattering response in some cases for the long wavelength limit. The use of reflection and transmission coefficients for the extraction of overall dynamic properties has received limited attention, e.g. in Fokin et al. (2007) and Zhu et al. (2012), as has the calculation of the impedance from field integrations (Yang et al., 2014). While the field averaging methods are inherently preferred in numerical analysis of complex 2D and 3D geometries and are mathematically more attractive, the scattering-based methods are more desirable as the natural foundation for measuring material properties in experiments due to the lack of access to micro-scale field quantities in the laboratory. It makes physical sense to weigh the suitability of averaging methods with the experimentally observable scattering response of finite samples. Therefore in this paper, overall properties of layered elastic and viscoelastic media are derived based on the scattering of finite samples from transfer matrix calculations. The method described here applies to symmetric and asymmetric unit cells equivalently and produces coupling parameter without any extra effort or special attention. To do so, it utilizes the full scattering response of a layered slab based on its transfer matrix, i.e. forward and backward solutions, even though in symmetric structures they are the same. It must be noted that the transfer matrix solution may be used to define point-wise impedance values, which match the ones derived from scattering analysis for the same structure. The general approach can be extended analytically or numerically to multi-dimensional structures, where a closed form solution may or may not be available. The issue of branch ambiguity for the phase velocity is discussed and resolved based on continuity considerations. The effect of energy conservation (in fully elastic cases) and passivity (in general) on the signs of the constitutive parameters are discussed briefly based on power flux considerations. The overall response of some asymmetric unit cell examples are derived next and followed by the results for an apparently asymmetric cell that would produce a symmetric infinitely periodic structure. A field averaging method based on direct integration of the wave equation is proposed and applied to the solutions from an infinitely periodic structure. The results match the scattering response for all frequencies with an equivalent unit cell. A comparison is made with another field averaging method, showing the compatibility in the long wavelength limit and deviations as one approaches the resonance areas. Finally, a short discussion of the applicability of these methods to higher dimensional structures is presented at the conclusion.

## 2. Transfer matrix of a slab with coupled constitutive law

The 1D elastodynamics homogeneous wave equation may be written as the combination of the continuity and equilibrium equations

$$\begin{aligned} v_{,x} &= \varepsilon_{,t}, \\ \sigma_{,x} &= p_{,t}, \end{aligned} \tag{1}$$

where $v$, $\sigma$, $p$, and $\varepsilon$ denote particle velocity, stress, momentum density (per unit volume), and strain, respectively, and subscripts after comma sign represent partial differentiation. Consider a Willis-type constitutive law

$$\begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} \eta & \kappa^{v\varepsilon} \\ \kappa^{\sigma p} & \mu \end{pmatrix} \begin{pmatrix} p \\ \varepsilon \end{pmatrix}, \tag{2}$$

where $\eta$ is the specific volume (inverse $\rho$, density) and $\mu$ is the relevant modulus of elasticity (e.g. longitudinal, shear, or Young's), while $\kappa^{v\varepsilon}$ and $\kappa^{\sigma p}$ denote particle velocity/strain and stress/momentum density couplings, respectively. Eq. (2) is a slightly transformed version of the formulation used in Milton and Willis (2007) and Willis (2009), in that stress is grouped with particle velocity instead of momentum density. Either of these forms may be derived from the other. Matrix inversion of Eq. (2) gives a more familiar form in terms of density and compliance. One reason for using this form is the similarity with Eq. (1). In frequency domain, the physical quantities described above ($\beta = v, \sigma, p, \varepsilon$) are written as $\beta(x,t) = \Re(\beta_c(x)e^{-i\omega t})$, where $\omega = 2\pi f$ is the angular frequency and $\beta_c$ is the complex amplitude. In the following, the subscript $c$ is dropped unless there is potential for confusion. The transfer matrix of a material layer normal to the $x$-axis, and identified by index $j$, relates the appropriate particle velocity and stress tensor components (in this case their $x$-components) on the two boundaries:

$$\begin{pmatrix} v(x_{j+1}) \\ \sigma(x_{j+1}) \end{pmatrix} = \mathsf{T}_j \begin{pmatrix} v(x_j) \\ \sigma(x_j) \end{pmatrix}. \tag{3}$$

$\mathsf{T}_j$ is a function of the frequency, the layer's material constitutive parameters, and its thickness $d_j = x_{j+1} - x_j$, where $x_j$ and $x_{j+1}$ represent the coordinates of the two surfaces of the layer. For each frequency value, the homogeneous wave equation has the general solution superposing two independent waves

$$\begin{pmatrix} v(x,k_j) \\ \sigma(x,k_j) \end{pmatrix} = \begin{pmatrix} 1 \\ -z_j(k_j) \end{pmatrix} A_j(k_j)e^{ik_jx}, \quad x_j \le x \le x_{j+1} \tag{4}$$

where the wave vector $k_j$ can take two possible values, say $k_j = k_j^+, k_j^-$, $A_j(k_j)$ are the complex amplitudes of the two waves, and

$$z_j(k_j) = -\frac{\sigma(x,k_j)}{v(x,k_j)}, \tag{5}$$

is the impedance of layer $j$ associated with the wave vector $k_j$. With this notation, and for a general case (suppressing the index) the transfer matrix may be written as

$$\mathsf{T}(\omega) = \frac{1}{z^+ - z^-} \begin{pmatrix} -z^-e^{ik^+d} + z^+e^{ik^-d} & -e^{ik^+d} + e^{ik^-d} \\ z^+z^-(e^{ik^+d} - e^{ik^-d}) & z^+e^{ik^+d} - z^-e^{ik^-d} \end{pmatrix}, \tag{6}$$

where for example $k^+ = k_j^+$, $k^- = k_j^-$, $z^+ = z_j(k_j^+)$, and $z^- = z_j(k_j^-)$. When $z^- = -z^+ = -z$ and $k^- = -k^+ = -k$, this will simplify to the more familiar form:

$$\mathsf{T} = \begin{pmatrix} \cos kd & -\frac{i}{z}\sin kd \\ -iz\sin kd & \cos kd \end{pmatrix}. \tag{7}$$