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Continuous gradient and discretized layered designs for control of stress wave scattering

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ABSTRACT

The scattering of stress waves in graded media is discussed in this paper using a conversion of the heterogeneous wave equation to a potential form. In this form, the scattering and inverse scattering methods of mathematical physics can be used with proper enforcement of the boundary conditions on the mechanical quantities, i.e. displacements and tractions. Two approaches for design of non-reflective media are analyzed, one for all frequencies with infinite thickness, and one with finite thickness for a target frequency range. In the latter approach, the transformed potential is set to zero, and the wave speed can have almost any desired profile, from which the profile of impedance is derived. The dispersive behavior of the finite thickness design is studied. Finally, an example is discussed in which a continuous non-reflective profile is discretized to a piece-wise constant profile and the effect of such unavoidable practical constraints on the scattering of the designed layer is shown to be minimal.

1. Introduction

Recent developments in the theory of micro-structured media particularly combined with advances in manufacturing complex shapes at various length scales (e.g. 3D printing, self-assembly) has led to design of devices and structures with multiple levels of heterogeneities. The interaction of waves in optics, electromagnetism, acoustics, and solid mechanics with a body that may be homogenized at the micro-scale, but has gradually changing properties, requires solutions of heterogeneous wave equation with constitutive properties that are potentially more complex than naturally occurring materials. The term “Transformation Optics” was coined after various researchers such as Leonhardt (2006) and Pendry et al. (2006) proposed cloaking and other scattering control mechanisms by transforming the wave equation using a coordinate system defined by the constitutive tensors. The concept has been extended to acoustics and elastodynamics by Milton et al. (2006), Norris (2008) and Norris and Shuvalov (2011). The practical applications have also been demonstrated, e.g. see Driscoll et al. (2006), Schurig et al. (2006), Wheeland et al. (2012) and Zigoneanu et al. (2014) and references therein. The analysis of wave propagation in heterogeneous media has a much longer history. The monograph by Brekhovskikh (1980) focuses on layered media, which are also of particular importance in earth sciences. One approach to address the mathematical difficulty of solving a PDE with spatially varying coefficients is to replace them with small perturbations on top of average values. This approach loses its utility away from the long wave length limit. An alternative is to change the spatial variables in such a way that the PDE ends up with constant coefficients but the heterogeneity is manifested in what mathematically may be considered an inhomogeneous source term for the PDE. In physics this is simply considered a potential. This approach has been utilized in this paper and in fact one can find the general structure in many classical sources, such as Brekhovskikh (1980). Such an approach is also very closely related to Transformation Optics/Acoustics formalism in the sense that the coordinate transforms are defined by the material constitutive parameters. Here, however, we focus on the appearance of the potential term. The transformed equation matches exactly with the time-independent Schrödinger equation, except for the fact that the boundary and jump conditions are naturally written for the mechanical quantities in the original equation. In the new, transformed, formulation, the new variables show up in the boundary and jump conditions in an elaborate form to match the physical requirement of the original quantities, i.e. displacement and traction. Nevertheless, the potential form enables the use of essentially the entire arsenal of theory of scattering in quantum mechanics. In this paper we focus on using certain elegant tools and solutions from inverse scattering theory, the main problem of which is to reconstruct the potential from the observed scattering response. The physical problem was discussed by Jost and Kohn (1952) and Levinson (1953) to relate the phase changes (considering the amplitude in spherically symmetric 1D

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conservative scattering is preserved) to the shape of the potential, while the mathematical problem was solved rigorously by Gel'fand and Levitan (1955) and Agranovich and Marchenko (1964). Extensive development of the subject is summarized by Faddeyev and Seckler (1963). In this paper we show how Marchenko's approach can be used to find non-reflective gradient designs for stress waves, similar to but also extending the one proposed for electromagnetic waves using dielectric layers by Kay and Moses (1956). This paper is structured as follows. First we discuss the formalism for transformation of heterogeneous wave equation to a potential form. Then, we discuss the all-frequency non-reflective design mentioned above. We then show the utility of the potential form by producing a family of finite thickness non-reflective designs, and explore the frequency dependence of scattering when the design is derived for a single central frequency point. Finally, we address the practical question of discretization of a continuous gradient design and its effect on scattering response.

2. Conversion of elastic 1D wave equation with spatial variation of coefficients to potential form

Consider a time-harmonic displacement $U(x,t) = \Re(u(x)e^{-i\omega t})$ and write the elastic 1D wave equation as,

$$L[u] = \frac{1}{\rho(x)}\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) + \omega^2 u = 0. \tag{1}$$

A general combined transformation, scaling both the dependent and independent variables u and x to ν and ξ, respectively, is written as

$$u(x) = \chi(x)\nu(x), \tag{2}$$
$$x = \varphi(\xi). \tag{2}$$

It is possible to find a transformation that would eliminate the first order derivative term in the resulting equation and make the coefficient of the second order derivative equal to one, while only adding a multiplicative potential term to the zeroth order term, as in the potential term in Schrödinger equation. It is actually possible to achieve this in two separate steps, first variation of the coordinate and then variation of parameter. To start use $x = \varphi(\xi)$ such that $dx/d\xi = c(x) = c(\varphi(\xi))$, where $(c(x))^2 = E(x)/\rho(x)$. This is essentially a switch to the characteristic coordinate. In the following, for the sake of simplicity, we take $c(x)$ to be real and positive. Although most of the following process may be carried out in the general case, many simplifications are enabled by this assumption. For more on such transformation without this conditions, see Brekhovskikh (1980). The new coordinate is then:

$$\xi = \varphi^{-1}(x) = \int^x \frac{dx'}{c(x')}. \tag{3}$$

In the following we tentatively assume that c and $Z = E/c = \rho c$ satisfy conditions to make this procedure applicable, except at a finite number of points. For example, we assume that c does not change its sign or vanish within the heterogeneous layer. Although it is possible to consider procedures to remove this restriction at a finite number of points (e.g. switching between the two characteristics to ensure ξ is monotonically increasing along with asymptotic or boundary analysis when $c \to 0$), we leave this analysis for specific designs. The continuity conditions on the impedance are slightly stronger as will be seen below. With this, the differential operator may be written as:

$$\begin{aligned} L[u] &= \frac{1}{\rho(x)}\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) + \omega^2 u \\ &= \frac{1}{\rho c}\frac{d}{d\xi}\left(\frac{E}{c}\right)\frac{du}{d\xi} + \frac{E}{\rho c^2}\frac{d^2u}{d\xi^2} + \omega^2 u \\ &= \left(\frac{1}{Z}\frac{dZ}{d\xi}\right)\frac{du}{d\xi} + \frac{d^2u}{d\xi^2} + \omega^2 u \\ &= \zeta(\xi)\frac{du}{d\xi} + \frac{d^2u}{d\xi^2} + \omega^2 u, \end{aligned} \tag{4}$$

where $\zeta(\xi) = dZ/Zd\xi = d\ln Z/d\xi$ is the logarithmic derivative of the impedance Z.[1] To remove the first order term, we now use variation of parameter:

$$L[u] = \zeta\chi'\nu + \zeta\chi\nu' + \chi''\nu + 2\chi'\nu' + \chi\nu'' + \omega^2\chi\nu = 0,$$

where $'$ represents differentiation with respect to ξ. Therefore, ν, satisfies

$$\frac{d^2\nu}{d\xi^2} + (\omega^2 - V(\xi))\nu = 0, \tag{5}$$

with

$$V(\xi) = -\frac{1}{\chi}(\chi'' + \zeta\chi')$$

provided that

$$2\chi' + \zeta\chi = 0.$$

Clearly any $\chi = CZ^{-1/2}$ is a solution of this equation, in which case

$$V(\xi) = \frac{2ZZ'' - Z'^2}{4Z^2} = \frac{d^2Z^{1/2}/d\xi^2}{Z^{1/2}}. \tag{6}$$

In the above derivation, a finite number of discontinuities in ζ should be treated separately. Specifically, at the material discontinuity boundaries, the continuity of displacement u and stress Edu/dx are transformed to the continuity of

$$Z^{-1/2}\nu, \tag{7}$$

$$Z^{-1/2}(Z\nu' - Z'\nu/2). \tag{8}$$

If Z and Z' are continuous and $Z \neq 0$, this is equivalent to continuity of ν', and the transformed quantities are ruled by the same conditions as one might find applicable to the physical potential form of the wave equation. If Z has discontinuity, they enforces finite jump conditions on ν and ν'.

The benefit of this transformation is that it makes possible the use of all the analytical and approximate methods devised in dealing with scattering from general potentials in physics and quantum mechanics. Inverse scattering methods have been routinely used to calculate the shape of potentials causing them, and therefore leading to information about quantum particles; See for example Kirst et al. (1989). In a design problem, desired scattering amplitudes and phases may be similarly used to derive the potential that will produce them. The above method allows constructing the profiles of constitutive properties compatible with such a potential and therefore the desired scattering response. The two distinctions to have in mind are (a) modified boundary and interface conditions and (b) the actual realizability of such designs in terms of extreme constitutive parameters that may be required, size of such structure, and deviations from the ideal response due to a discretized, piece-wise constant fabrication. The former issue is only an additional mathematical step, while the latter is studied in the following examples.

[1] Since we are using a harmonic form, all results apply to complex-valued Z. Alternatively, to limit to real-valued constitutive parameters, one may simply use forms such as $\ln|Z|$ or $|Z|^{-1/2}$, whenever necessary.