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Furthest reach of a uniform cantilevered elastica

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ABSTRACT

A uniform elastic cantilever is subjected to a uniformly distributed load or a concentrated load at its tip. The angle of the fixed end with the horizontal is varied until the maximum horizontal distance (projection) from the fixed end to the horizontal location of the tip is attained. The beam is modeled as an inextensible elastica, and numerical results are obtained with the use of a shooting method. For the optimal solution (furthest reach), the tip is below the level of the fixed end. Experiments are conducted to verify the analysis for a heavy cantilever (i.e., only subjected to its self-weight).

1. Introduction

A uniform, massless, cantilevered elastica with a concentrated load at its tip was considered by Wang [1] and Batista [2]. For a given value of the tip load, the angle of the fixed end with the horizontal was computed for which the tip was at the same level as the fixed end. The associated horizontal span was termed the "longest reach."

In the present study, the restriction of equal end heights is removed, and the maximum horizontal distance (projection) between the fixed end and the tip is determined (see Fig. 1). Following the terminology of Wei et al. [3], this distance will be called the "furthest reach." The problem involves optimization of the angle of the fixed end. A tip load is considered, and also a uniformly distributed load that could be the self-weight of the beam (i.e., a "heavy beam"). In nondimensional terms, results for the optimal angle, furthest reach, and corresponding tip deflection are presented.

Other papers analyzing large deflections of cantilevers under such loads include Wang [4], Beléndez et al. [5], Wang et al. [6], and Kimiaeifar et al. [7]. Applications of the problem considered here include sensing of objects, with the use of flexible 'whiskers' (Zhao and Rahn [8]; Zhu et al. [9], Iida and Nurzaman [10]; Lucianna et al. [11]), cantilevers as in atomic force microscopy (AFM) (Bausells [12]), and continuum manipulators (Gao et al. [13]).

McMahon [14,15] was interested in the lengths of tree branches. He conducted experiments in which uniform rubber cylinders of different lengths were cantilevered horizontally and the length yielding the furthest reach was sought. In McMahon [16], the slope (angle) of the cantilever at the fixed end was varied. An analysis was carried out. For any given slope, again the beam length that maximized the lateral reach was determined. The problem treated here is different, with the length fixed and the optimal angle computed.

For the case of zero slope at the fixed end, Wei et al. [3] found the optimal distribution of material giving the furthest reach of a heavy beam. At the tip, the optimal cross-sectional area was zero and the beam pointed vertically downward. Plaut and Virgin [17] treated a related problem, varying the material distribution and minimizing the vertical tip deflection; a minimum cross-sectional area was specified.

2. Formulation

The beam has length L and constant bending stiffness EI. Points on the beam have coordinates $X(S)$ and $Y(S)$, and rotation $\theta(S)$ with respect to the X axis, where S is the arc length from the fixed end. The bending moment is $M(S)$. Including both the tip load P and uniformly distributed load W, the vertical internal force is $P+(L-S)W$. The analysis is conducted in terms of the nondimensional quantities $\theta(s)$ and

$$x=X/L, y=Y/L, s=S/L, m=ML/(EI), p=PL^2/(EI), w=WL^3/(EI) \tag{1}$$

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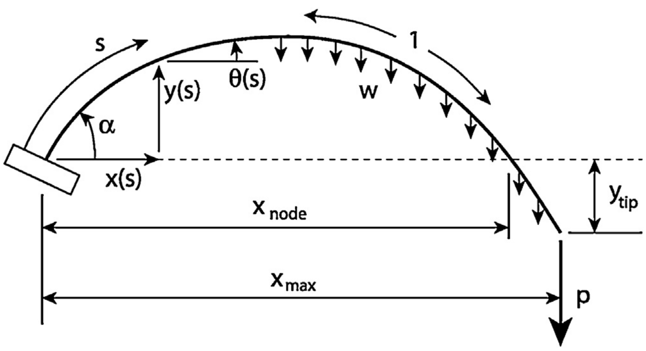

Fig. 1. Schematic of beam in optimal configuration.

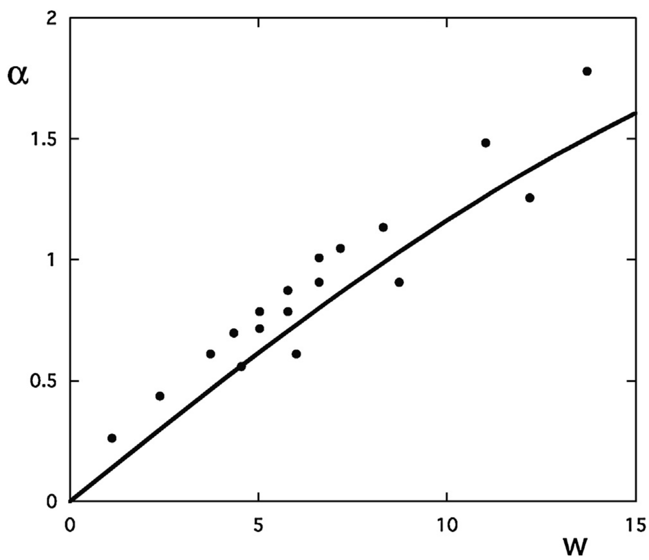

Fig. 2. Optimal angle α as function of w.

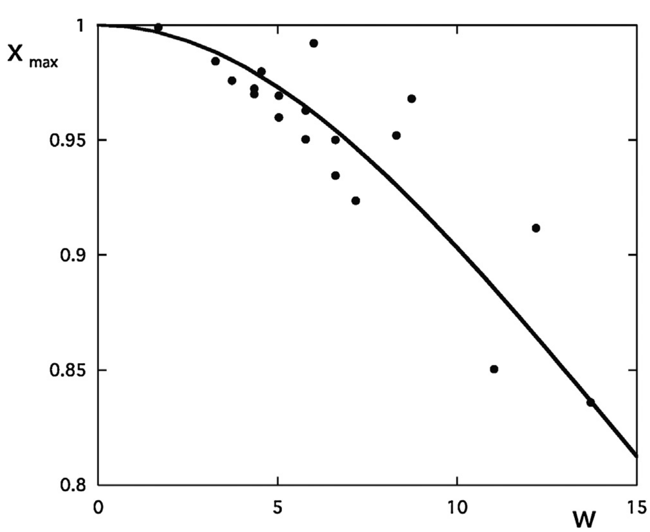

Fig. 3. Furthest reach $x_{\max}$ as function of w.

Fig. 1 shows the elastica in terms of these quantities. The angle $\theta(0)$ at the fixed end is denoted α. The governing equations are [18]

$$x' = \cos\theta, y' = \sin\theta, \theta' = m, m' = [p + (1-s)w]\cos\theta \tag{2}$$

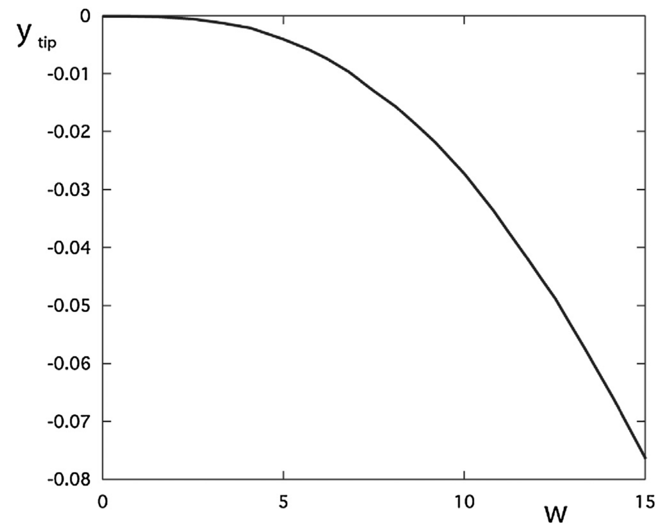

Fig. 4. Tip deflection y_{tip} as function of w.

Table 1
Numerical results for cantilever with uniformly distributed load.

w	α	x_{max}	y_{tip}	x_{node}
0	0	1	0	–
2.5	0.3110	0.9931	−0.0005	0.9885
5	0.6130	0.9731	−0.0040	0.9542
7.5	0.8983	0.9422	−0.0127	0.9025
10	1.1615	0.9032	−0.0273	0.8401
12.5	1.3976	0.8590	−0.0492	0.7702
15	1.6060	0.8124	−0.0767	0.7001

The boundary conditions are $x(0) = 0$, $y(0) = 0$, $\theta(0) = \alpha$, and $m(1) = 0$. Numerical solutions are obtained using a shooting method with Mathematica. The quantities p, w, and α are specified, and the subroutines NDSolve and FindRoot are applied to vary $m(0)$ until $m(1) = 0$. The corresponding value of $x(1)$ is found from the solution. Then α is varied and the procedure is repeated until the maximum value of $x(1)$, called $x_{\max}$ (i.e., the furthest reach), is attained.

When $x(1) = x_{\max}$, it turns out that the tip of the cantilever is below the level of the fixed end, i.e., $y(1) < 0$, and this value of $y(1)$ is denoted y_{tip}.

3. Results for uniformly distributed load

The case $p = 0$ (no tip load) is considered in this section. Numerical results are presented as solid curves in Figs. 2–4 for the range $0 \le w \le 15$. In Fig. 2, the angle α that yields the furthest reach is plotted as a function of the load (or self-weight) w. The slope of the curve decreases slightly as w increases. For the massless, unloaded cantilever ($w = 0$), the solution is the straight horizontal beam, with $\alpha = 0$, $x_{\max} = 1$, and $y_{tip} = 0$. An approximation for the optimal angle is given by

$$\alpha \approx 0.1264\,w - 0.0004868w^2 - 0.00005376w^3 \tag{3}$$

The error in the approximation is less than 0.002 rad for the range in Fig. 2.

In Fig. 3, the furthest reach $x_{\max}$ is plotted as a function of w. The slope of the curve decreases as w increases. Finally, in Fig. 4, y_{tip} is plotted as a function of w. Again the slope of the curve decreases as w increases.

Numerical values are listed in Table 1 for $w = 0$, 2.5, 5, 7.5, 10, 12.5, and 15. The quantity x_{node} in the last column is the value of x (within the span) where $y = 0$, as shown in Fig. 1.