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A study of the critical strain of hyperelastic materials: A new kinematic frame and the leading order term

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ABSTRACT

In this paper, an investigation is made on the problem of critical strain of hyperelastic materials. A new kinematic frame, which takes into account the characteristics of the material, is firstly proposed to consider a hyperelastic rectangular layer under compression. The simplified model equations are derived with the aid of virtual principal and asymptotic expansion method. Through linear bifurcation analysis, the critical strain is determined, which is in good agreement with existing results. Comparisons between the classical *Eular Beam theory*, *first-order shear deformation theory* and the present frame are also made, and it shows that the present frame can provide much better results than the other two frames. Moreover, under the same framework, several other typical compressible hyperelastic materials are examined. One interesting finding is that their critical strain share the same leading order term which is independent of the form of strain energy functions.

1. Introduction

Critical strain of nonlinear elastic materials induced by uniaxial compressions is a classical topic in nonlinear elasticity. In general, the critical strain can be determined by linear bifurcation analysis. In the early days, there were many efforts which devoted to linear bifurcation analysis of either linear or nonlinear bars, rods and beams, such as [1–5]. These works provided nice results for the critical bifurcation loads and eigenfunctions. Particularly, there were some works which deal with nonlinear materials during the past several years. Merodio and his collaborators studied the bifurcation of the cylinder under different boundary conditions in recent years. In [6], Rodrígue and Merodio investigated the bifurcation of arterial walls of patients with Marfan's syndrome by modeling the arterial as a circular cylinder with finite wall thickness. The effects of the wall thickness have been studied. Under combined axial loading and internal pressure condition, the same authors [7] studied the bifurcation of a membrane cylinder for three different modes. In [8], Rodrígue and Merodio analyzed the buckling and postbuckling of residually-stressed elastic body by finite element method. The bifurcation points and postcritical behavior are obtained. In [9], within a two-dimensional context, Roxburgh and Ogden obtained the critical stress of a two-dimensional layer by linear bifurcation analysis. In [10], the authors applied couple series and asymptotic expansions method to study the bifurcation and post-bifurcation solutions of a nonlinear hyperelastic layer, both critical strain and the post-bifurcation analytical solutions are obtained. In [11], the authors considered the post-bifurcation solutions of a hyperelastic rectangular block under compression/tension in a plane-strain setting, and they obtained the leading order term of the critical strain and the post-bifurcation solutions for small aspect ratio. Though there are numerous contributions to the critical strain of separate hyperelastic materials, we find that there are few results for the comparisons among critical strain of different materials.

As a starting point to determine the critical strain, one usually makes some assumptions on the form of the displacement field. For linear elastic material, there are two classical displacement field assumptions: Euler and Timoshenko beam theory. In [12,13], by taking into account the effect of the shear stress, Waas applied the first order asymptotic expansions to approximate the displacement of the beam. More higher-order shear beam models can be found in [14]. However, the aforementioned theories and models may not work for hyperelastic materials, as one can check it from [10,11]. This might be due to the order of accuracy or the absence of material parameters in the displacement assumptions. To this end, in this paper we present a new kinematic frame for hyperelastic materials. Based on the first-order shear deformation theory (FSDT), we develop a New FSDT(NFSDT) to describe the displacement field. The

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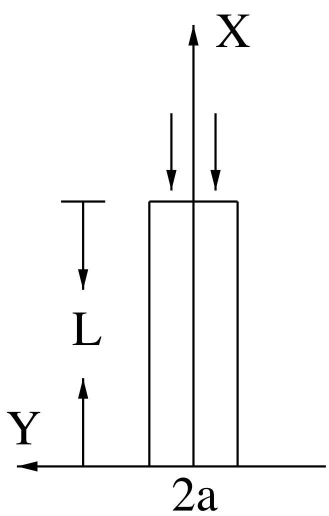

Fig. 1. The geometry of layer.

NFSDT is based on a more rigorous kinematics of displacements, and takes into account the shear stress effect and characteristics of the material. It differs from other FSDTs and existing higher order theories in the inclusion of the effect of characteristics of hyperelastic materials. Moreover, as one will see from our latter derivations, it turns out that the NFSDT can provide a unified and more simpler frame for the calculations of critical strain of hyperelastic materials. For the problems considered in this paper, with NFSDTs the results agree well with the existing results in the open literature. Thus, with a new kinematic frame a study of critical strain of different hyperelastic materials is carried out in this work.

The remaining part of this paper is arranged as follows. In Section 2, we propose a new kinematic frame for two-dimensional rectangular layer, and obtain two nonlinear coupled ordinary differential equations as model equations with the aid of asymptotic expansions and virtual principle. In Section 3, the critical stain is obtained by linear bifurcation analysis. In Section 4, we apply the same model to determine the critical strain for several kinds of hyperelastic beams, and find that the critical strains of the compressible hyperelastic materials share the same leading order term. Some conclusions are drawn in Section 5.

2. Model for a two-dimensional layer

In this section, we firstly present a new kinematic frame for the two-dimensional hyperelastic layer, and then use the asymptotic expansions method and the variational principle to derive the model equations for the layer.

We consider a two-dimensional compressible hyperelastic moderately thin layer with initially unstressed state, which has a thickness $2a$, length L. (X, Y) denotes the Cartesian coordinates of the material points in the reference configuration (see Fig. 1).

2.1. A new kinematic frame

Generally, the equilibrium field equations can be expressed in terms of the displacement of the layer, and a lot of simplified models are derived based on the approximations (expansions) of the displacement. According to the traditional small-strain *Euler–Bernoulli Beam Theory*, the displacement of the layer can be written as

$$\begin{cases} U_L(X,Y) = U(X) - YV_X(X), \\ V_L(X,Y) = V(X), \end{cases} \tag{2.1}$$

where $U_L(X, Y)$, $V_L(X, Y)$ are the displacements of a material point in direction X and Y, respectively. $U(X)$, $V(X)$ are the displacements of the centroidal line in direction X and Y, respectively. Hereafter, the symbol f_X represents the derivative of the function f with respect to X. In [12,13], Waas applied the first order asymptotic expansions to approximate the displacement of the beam as

$$\begin{cases} U_L(X,Y) = U(X) - YV_X(X), \\ V_L(X,Y) = V(X) + YU_X(X). \end{cases} \tag{2.2}$$

In [10], Dai applied the coupled series expansions to simplify the displacement of the layer as

$$\begin{cases} U_L(X,Y) = U(X) + YU_1(X) + Y^3U_2(X) + \cdots \\ +\delta(Y^2U_3(X) + Y^4U_4(X) + \cdots), \\ V_L(X,Y) = \delta(V(X) + YV_1(X) + Y^3V_2(X) + \cdots) \\ +Y^2V_3(X) + Y^4V_4(X) + \cdots, \end{cases} \tag{2.3}$$

where the functions $U_i(X)$, $V_i(X)$ are unknown functions, and δ is a parameter.

Formulae (2.1) and (2.2) are valid for linear elastic materials. However, since these formulae neglect shear strain effects which are very important for the post-buckling behavior, they are invalid for nonlinear hyperelastic materials. Expansion (2.3) is valid for hyperelastic materials and has been successfully applied to deal with some important problems, but this formula would lead to very long calculations.

In this paper, we only consider the case in which instability initiates when the load is in the neighborhood of the critical load. Under this assumption, the post-bifurcation state would be close to the pre-bifurcation uniform deformation state. In order to capture the bifurcation state of the layer, we present a new leading order asymptotic model to describe displacement field of the layer in finite deformations as

$$\begin{cases} U_L(X,Y) = U(X) - YV_X(X), \\ V_L(X,Y) = V(X) - \nu YU_X(X) + \dfrac{1}{2}\nu Y^2V_{XX}(X). \end{cases} \tag{2.4}$$

Here, the parameter ν is the in-plane Poisson's ratio. This new formula is expected to provide more accurate results for the displacement of the layer, since it takes into account the effect of the material parameter ν and the shear strain.

Remark. For the in-plane uniform deformation of hyperelastic materials, (2.4) are exactly the same as

$$\begin{cases} U_L(X,Y) = U(X), \\ V_L(X,Y) = -Y\nu U_X(X), \end{cases} \tag{2.5}$$

where $U(X) = k\,X$, k is a constant.

Before using formula (2.4), we need to determine the parameter ν. To this end, we first consider a two dimensional plane under in plane axial compression (see Fig. 1). For the pre-bifurcation state of the plane, the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} k & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

where k, m are the principal stretches in direction X, Y, respectively. They are dependent upon the solid's constitutive response and determined by

$$\frac{\partial \Phi}{\partial m} = 0, \tag{2.7}$$