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Research paper

Unique minimum norm solution to redundant reaction forces in multibody systems

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ABSTRACT

Multibody system models that rely on the rigid body assumption are used in many engineering applications. In such models redundant constraints often appear, which is caused by a lack of information about the physical system. This prevents the unique determination of all reaction forces associated with the constraints. Sometimes the resultant effect of these redundant constraints can be considered instead of computing individual constraint reactions. When, on the other hand, individual reactions are key, additional assumptions and methods have to be considered. This paper analyzes the conditions under which the minimum norm solution to this problem is non-unique, provides a physical interpretation of the issue, proposes a simple method to normalize the minimum norm solution and make it unique, and extends these principles to unilaterally constrained multibody systems. The theoretical development and validation are carried out on the basis of meaningful examples, namely, two pendulum-type systems, a box on a plane, a one-degree-of-freedom mechanism, and a 33-degree-of-freedom chain.

1. Introduction

Kinematic constraints (both *bilateral* and *unilateral*) are one of the main tools to represent physical interactions such as joints in mechanical systems. In this way, the interaction forces are modeled as constraint reaction forces, which in turn can be represented through the so-called Lagrange multipliers. Even though Lagrange multipliers have traditionally been used on a mathematical basis, a physics-based derivation that shows their reaction force nature is also possible. The physical unit and interpretation of a Lagrange multiplier depend on how the associated constraint is defined. In the general sense, constraints that represent physical interactions are essentially “constrained generalized velocities” [1] that are either integrable (*holonomic*) or non-integrable (*nonholonomic*). A Lagrange multiplier is the force associated with a constrained generalized velocity, and their product must represent power.

Let us consider a set of holonomic, bilateral constraints. A very common assumption within rigid body dynamics is that bilateral constraints are independent. The reason is that the presence of redundant (dependent, compatible) constraints results in an overconstrained system. In such a case, the reaction forces are not unique because the rank of the Jacobian matrix of constraints is less than the number of constraints, which means that there is an infinite number of possible solutions to the problem. In frictionless models, the forward dynamics will not be affected by this phenomenon, because its solvability does not require a full rank Jacobian matrix.

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Nevertheless, the lack of full rank of the Jacobian matrix does have an effect on the determination of the reaction forces in the constraint subspace. One approach is to find the so-called *minimum norm solution*, which is the solution with the smallest Euclidian norm among the infinite possible solutions. The minimum norm solution can be computed using the Moore–Penrose pseudoinverse of the Jacobian matrix, which is a unique generalized inverse often referred to as *pseudoinverse*. The physical interpretation of the minimum norm solution is that self-equilibrating constraint forces are neglected, which is a reasonable assumption in the absence of information about pre-loads, assembly defects and thermal stresses [2,3]. A number of authors have used this solution to compute redundant reaction forces in different scenarios. For instance, the pseudoinverse has been used to calculate redundant muscular forces within biomechanical models [4,5].

The minimum norm solution, however, appears to be non-unique. One of the reasons is that when the constraint equations carry different units, the transformation of units will also change the values of the reaction forces. Even if the constraints do not have mixed units or are unitless, there are different ways to formulate them, each of which will imply a different Jacobian matrix and different reaction values. Some authors introduce weighting so that the associated minimum norm solution resembles the solution obtained through constraint flexibility [3], or so that certain optimization goals are achieved [6]. However, this requires additional information about the structural properties of the system beyond the rigid body assumption, and it is likely more computationally expensive than the constraint relaxation method.

There are three other methods to determine redundant reaction forces: constraint elimination, constraint relaxation (penalty methods and compliant constraints) [7–10], and flexible multibody methods [11–13]. Further, an approach to separate redundant from non-redundant joint constraints was also presented [14,15], including joints with Coulomb friction [11] and nonholonomic constraints [16]. The authors in [12] argue that only the full consideration of the flexibility of bodies and possibly joints gives a unique solution to the reaction forces, at the cost of using more complex formulations for flexible multibody systems and a great deal of additional information about the strength of materials. According to them, all other approaches (including partial flexibility [17]) would suffer from multiplicity of solutions.

Constraint redundancy is handled by commercial software packages in various different ways. For instance, some of them detect redundant constraints and remove them from the analysis. This changes the physics of the problem by eliminating the corresponding reaction forces altogether. Other packages relax all constraints through heuristic compliance coefficients, which can benefit performance and stability but may hinder physical accuracy.

Very little work in the literature appears to address the use and interpretation of the minimum norm solution for unilaterally constrained contact problems. The authors in [18] presented a least-squares solution to redundant impact problems, and argued that such a solution is more stable and more energetically sensible for rigid systems than those obtained through constraint elimination and fully compliant models. This is consistent with our observations, and encouraged us to algebraically, geometrically and physically explore the meaning of the minimum norm solution within the contact dynamics area.

In summary, certain publications advocate for the inclusion of structural properties as the only way to obtain “unique” or “mechanically consistent” reaction forces. This paper, on the other hand, presents a method for the computation of unique reaction forces that fully respects the physics of the original problem without requiring additional information about structural properties. We will show that this can be a very reasonable strategy for models with both bilateral and unilateral constraints.

2. Bilateral constraints

Because rigid body assumptions are not only representative, but also efficient and accurate in numerous mechanical systems, the analyst often opts for constraint elimination or minimum norm solutions to the redundancy problem. When all reaction forces, including the redundant ones, must be calculated, the minimum norm solution is the alternative of choice. Hence, the circumstances under which the minimum norm solution is unique are of great relevance.

2.1. Dynamic equations

Consider a set of holonomic, bilateral constraints, which can be expressed in terms of the generalized coordinates of the mechanical system and time:

$$\boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0} \tag{1}$$

where $\boldsymbol{\Phi}$ is the $m \times 1$ array of bilateral constraints and $\mathbf{q}$ is the $n \times 1$ array of generalized coordinates. The forces corresponding to the constraints can be introduced into the dynamic equations as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{f}_{\mathrm{a}} \tag{2}$$

where $\mathbf{A} \equiv \boldsymbol{\Phi}_{\mathbf{q}}$ is the $m \times n$ Jacobian matrix of constraint equations, r is its rank, $\boldsymbol{\lambda}$ is a set of m Lagrange multipliers, $\mathbf{M}$ is the $n \times n$ generalized mass matrix, and $\mathbf{f}_{\mathrm{a}}$ is the $n \times 1$ array of applied and velocity-dependent inertial forces. The value of the Lagrange multipliers can be found using the following linear system of equations:

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{f}_{\mathrm{c}} \tag{3}$$

where $\mathbf{f}_{\mathrm{c}} = \mathbf{f}_{\mathrm{a}} - \mathbf{M}\ddot{\mathbf{q}}$ is the array of *generalized constraint forces*.