Mechanism and Machine Theory 117 (2017) 230–243

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmachtheory

Research paper

Complexity of the forward kinematic map

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ARTICLE INFO

Article history:
Received 5 January 2017
Revised 10 July 2017
Accepted 21 July 2017

Keywords:
Robot arm
Configuration space
Working space
Forward kinematics
Robust motion planning
Topological robotics

ABSTRACT

The main objective of this paper is to introduce a new method for qualitative analysis of various designs of robot arms. To this end we define the complexity of a map, examine its main properties and develop some methods of computation. In particular, when applied to a forward kinematic map associated to some robot arm structure, the complexity measures the inherent discontinuities that arise when one attempts to solve the motion planning problem for any set of input data. In the second part of the paper, we consider instabilities of motion planning in the proximity of singular points, and present explicit computations for several common robot arm configurations.

1. Introduction

In this paper we introduce and discuss a new qualitative measure of the complexity of a forward kinematic map from the configuration space of the robot arm joints to the working space of the end-effector. Let us illustrate the problem on a familiar example of a robot arm with n revolute joints. The position of the ith joint is uniquely determined by some angle of rotation θ_i, so we may identify the position of each joint with a point on the unit circle T, and the combined position of all n joints with an n-tuple of values $(\theta_1, \ldots, \theta_n) \in T^n = T \times \ldots \times T$ (n factors). The position of the end-effector is determined by the spatial location and the orientation, so it corresponds to a point $(\vec{r}, R)$ in the cartesian product $\mathbb{R}^3 \times SO(3)$ (here we identify the space of all possible orientation of a rigid body with the set $SO(3)$ of all orthogonal 3×3 matrices of determinant 1). The exact form of the resulting forward kinematic map $F : T^n \to \mathbb{R}^3 \times SO(3)$ depends on the lengths and the respective inclinations of the axes, and is usually given in terms of Denavit–Hartenberg matrices but the explicit formulae will not be relevant for the main part of our discussion.

The motion planning problem in this setting may be stated as follows: given an initial state of joint parameters $(\theta_1, \ldots, \theta_n) \in T^n$ and a required end-effector position $(\vec{r}, R) \in \mathbb{R}^3 \times SO(3)$, find motions of the joints starting at $(\theta_1, \ldots, \theta_n)$ and ending in a position of joints such that the corresponding end-effector position is $(\vec{r}, R)$. The problem may be modeled as follows: let $P(T^n)$ denote the space of all possible paths in T^n (i.e., continuous maps α: $[0, 1] \to T^n$ from the interval $[0, 1]$ to the joint parameter space), and let $\pi : P(T^n) \to T^n \times (\mathbb{R}^3 \times SO(3))$ be the map that to each path assigns its starting and ending position, $\pi(\alpha) := (\alpha(0), F(\alpha(1)))$. Then the solution of the motion planning problem can be viewed as an inverse map $\rho : T^n \times (\mathbb{R}^3 \times SO(3)) \to P(T^n)$, with the property $\pi(\rho((\theta_1, \ldots, \theta_n), (\vec{r}, R))) = ((\theta_1, \ldots, \theta_n), (\vec{r}, R))$. We are mainly interested in robust motion plans, such that a small perturbation of the initial data results in a comparatively small perturbation of

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[1] The author was partially supported by the Slovenian Research Agency grant P1-02920101.

http://dx.doi.org/10.1016/j.mechmachtheory.2017.07.015
0094-114X/

the corresponding motion plan. In other words, we normally require that the map ρ is continuous with respect to the input data.

The starting point of our investigation is the following fundamental observation (see Theorem 2.1):

If there exists a robust global solution of the motion planning problem for the map F, then there also exists a continuous global solution of the inverse kinematic problem for the map F.

Recall that the inverse kinematic problem requires to find the values of the joint positions given the position and orientation of the end-effector (see for example [[26], Section 1.7] and [[2], Section 11.3]). It is known that the inverse kinematic map can be computed in closed form only for a very restricted class of simple manipulators. Moreover, since the kinematic map is rarely injective, the solution to the inverse kinematic problem usually involves some choice between several possible pre-images of the given end-effector position, so the resulting inverse kinematic map is not continuous. It follows that a solution of the motion planning problem will almost always require a partition of the input data space into smaller domains, over which a robust motion plan can be constructed. The minimal number of domains that is needed to cover all possible input data measures the complexity of the motion planning for a given robot arm configuration. We are going to describe a mathematical model that will allow a clear definition of the complexity and develop several methods for its computation.

1.1. Prior work

Motion planning is one of the basic problems in robotics, it has been extensively studied, and there exists a vast body of literature on the topic. We are going to rely on Hollerbach [16] and Kavraki and LaValle [18] as basic references. In the standard formulation of the motion planning problem one specifies the *configuration space* $\mathcal{C}$ of the robot device (which normally correspond to the set of joint parameters but may also take into account various limitations), the *working space* $\mathcal{W}$ (spatial position and orientation that can be reached by the robot), and *obstacle regions* in $\mathcal{W}$. Then one considers *queries* that consist of an *initial configuration* $q_I \in \mathcal{C}$ and a *goal configuration* $q_G \in \mathcal{C}$ (corresponding to the desired position and orientation of the end-effector), and asks for a path $\rho : [0, 1] \to \mathcal{C}$ that avoids obstacles, and for which $\rho(0) = q_I$ and $\rho(1) = q_G$.

The complexity of motion planning was mostly considered within the context of computational complexity: indeed, the search for explicit algorithms aimed at the solution of a given motion planning problem was often accompanied by more general considerations regarding the algorithmic complexity of various solutions – see [1,23]. For a more recent study that extensively uses topological methods and is similar in spirit to our approach see [5].

A more geometrical measure for the complexity of motion planning was introduced by Farber [6] who observed that algorithmic solutions normally yield robust motion plans. He then defined the concept of the *topological complexity* of motion planning in the configuration space of a mechanical device as the minimal number of continuous partial solutions to the motion planning problem. In many cases the computation of the topological complexity of a space may be reduced to the computation of a very classical numerical invariant, called the Lusternik–Schnirelmann category. Indeed, in certain sense the study of the complexity of the geometric motion planning can be traced back to the1930s, to the work in variational calculus by L. Lusternik and L. Schnirelmann. They introduced what is today called the *Lusternik–Shnirelmann category* of a space, denoted $\mathrm{cat}(X)$, as a tool to estimate the number of critical points of a smooth map. Their work was widely extended both in analysis, most notably by Schwartz [24] and Palais [22], and in topology, by Fox [9], Ganea [10], James [17] and many others. Today Lusternik–Schnirelmann category is a well-developed theory with many ramifications and methods of computation techniques that allow for the systematic determination of the category for most of the spaces that will appear in this paper. It is interesting to see how this very classical and independently developed theory found its application in the study of problems in robotics. For the convenience of the reader we will give formal definitions of the Lusternik–Schnirelmann category and of Farber's topological complexity in Section 3. The standard reference for the Lusternik–Schnirelmann category is [3], for an overview of principal results on topological complexity see [8], for a less technical and very readable account, see also [[13], Chapter 8].

The study of the complexity of a map is a natural continuation of Farber's work and was suggested as a problem by A. Dranishnikov during the conference on Applied Algebraic Topology in Castro Urdiales (Spain, 2014). In spite of strong formal similarities, the flavor of this new theory is quite different from the topological complexity. A partial explanation can be found in some earlier work by Hollerbach [15] and Gottlieb [11], who studied the possibility to avoid singularities of the forward kinematic map by introducing additional joints. They proved that under standard assumptions a forward kinematic map always has singularities and (with rare exceptions) does not admit global inverse kinematics. As a consequence, the study of the complexity of a map seems to be less amenable to purely homotopy-theoretical methods.

1.2. Our contribution

We introduce a general framework for the study of the complexity of a continuous map. Roughly speaking, the complexity of a map $F : \mathcal{C} \to \mathcal{W}$ is the minimal number of robust rules that take as input pairs of points $(c, w) \in \mathcal{C} \times \mathcal{W}$, and yield paths $\rho = \rho(c, w)$ starting at c and ending at some c' that is mapped by F to w. We first show that under the assumption that F is regular and admits a right inverse (section), the computation of the complexity of F can be reduced to the topological complexities of the spaces $\mathcal{C}$ and $\mathcal{W}$ in the sense of Farber. However, a forward kinematic map usually satisfy these assumptions only locally, which causes considerable difficulties and requires the development of entirely new methods. As