Mechanism and Machine Theory 114 (2017) 11–19

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmachtheory

Research paper

Algebraic analysis of overconstrained single loop four link mechanisms with revolute and prismatic joints

Martin Pfurner*, Thomas Stigger, Manfred L. Husty

Unit Geometry and CAD, University of Innsbruck, Austria

ARTICLE INFO

Article history:
Received 15 December 2016
Revised 28 March 2017
Accepted 28 March 2017

Keywords:
Bennett mechanism
RPRP mechanism
Four link mechanism
Algebraic analysis
Kinematic mapping

ABSTRACT

Although spatial four link single loop mechanisms with revolute and prismatic joints are well investigated in many publications, there are only few of them dealing with the geometric conditions that cause these generally rigid structures to become movable. This article deals with the derivation of the necessary geometrical design conditions for mobility using kinematic mapping. Furthermore, the coupler motions of the mechanisms are computed. For the Bennett mechanism and the overconstrained *RPRP* chain the ruled surfaces representing the base surfaces for the line symmetric motions as well as the moving and fixed axodes are computed using the derived design parameters only. All surfaces are computed explicitly and shown in examples. A further contribution of this article is that in case of the *RPRP* mechanism algebraic mobility conditions in the design parameters are found which have never been published before.

1. Introduction

Spatial four link closed single loop mechanisms having either revolute or prismatic joints are good examples of closed kinematic chains that are, in general, rigid structures. Only with special geometric design of the linkages it is possible that the mechanisms become mobile. The study of these mechanisms is a well investigated field in kinematics, but most of the publications deal with motion analysis and related topics rather than with the derivation of mobility conditions. It is remarkable that among the vast publications dealing with spatial overconstrained four bar linkages only very few present the derivation of necessary and sufficient design conditions for mobility. It even seems that a complete list of such algebraic conditions is missing.

The aim of this paper is to compute the geometric relations for the design such that the four link closed single loop mechanisms with revolute and prismatic joints have mobility one. In 1922 Delassus [1] studied all the four-bar linkages with helical, revolute or prismatic joints. Waldron [2,3] recalled these linkages but for the cases addressed in this paper an explicit solution of necessary and sufficient conditions for mobility is not given in either publication. Krames [4,5] published a series of papers in 1937 dealing with line symmetric motions using purely synthetic methods. He reveals in remarkably deep geometric discussions that the coupler motion of a Bennett mechanism is a line-symmetric motion with a hyperboloid as basic surface. Some of the results of this series of papers are also presented in Bottema and Roth [6, CH. 9, Section 7]

The most famous example of this type of closed chains is the Bennett mechanism, which is a closed single loop chain with four revolute joints. It was introduced by Bennett [7] in 1903 as a mechanism, where the common normals of the

* Corresponding author.
E-mail address: Martin.Pfurner@uibk.ac.at (M. Pfurner).

http://dx.doi.org/10.1016/j.mechmachtheory.2017.03.014
0094-114X/

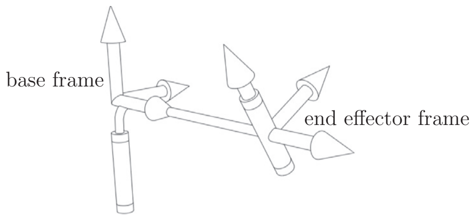

Fig. 1. Canonical 2*R* chain with its base and moving frame.

axes form a skew isogram. This mechanism was analyzed extensively under different viewpoints (see for example [8–14]) and used in various combinations, for example to construct overconstrained 6*R* linkages as summarized in [15]. Karger [16] presented a method in 1994, different to Delassus, to show that this is the only spatial four link chain with only rotational joints that is mobile.

Another known example of four link closed loop chains is the *RPRP*-mechanism. Grünwald [17] showed a mechanism that can also be realized as an *RPRP*. This chain was extensively analyzed from different viewpoints for example in [18–21]. In [2,20] it is stated that the axes of the prismatic joints have to be symmetric with respect to the plane containing the parallel revolute axes. Geometrically this is correct but when building such a mechanism it is possible to make mechanisms that do not have this property. In this paper we will derive algebraic conditions on the Denavit-Hartenberg (DH) [22] parameters proving that the *RPRP* mechanism does not have to be built symmetrically.

The investigation of overconstrained 4-link mechanisms is important because both types of these mechanisms have been used recently as building blocks of metamorphic linkages [23–26].

In contrast to the aforementioned publications we want to give a purely algebraic proof that the Bennett and the *RPRP* mechanism are the only movable four bars with this kind of joints. Therefore we discuss all possible four link mechanisms having revolute and/or prismatic joints and explicitly derive constraints on the design such that these mechanisms are movable. Additionally we compute the motion of the coupler system, the base surface for the line symmetric motion and the moving and fixed axode of the motion for the first time using only Denavit Hartenberg parameters to describe them.

The paper is organized as follows. In Section 2 at first the mathematical background is given and then all possible four link chains are analyzed, conditions for mobility are derived and the coupler motions of the mobile chains are discussed using the DH-parameters only. A conclusion is given in Section 3.

2. Analysis of single loop four link mechanisms

In this article closed single loop mechanisms are analyzed in kinematic image space P^7 using the methods presented in [27–30]. The homogeneous coordinates in P^7 are denoted by $(x_0\colon x_1\colon x_2\colon x_3\colon y_0\colon y_1\colon y_2\colon y_3)$ and the design of the mechanisms is given in Denavit Hartenberg convention. The analysis is performed by the intersection of constraint manifolds in P^7. The four link mechanisms are divided into two open serial two link mechanisms. In order to achieve overconstrained mechanisms the constraint manifolds of these sub chains must not only intersect in discrete points, they have to intersect at least in a curve. It was shown in [27,31] that the constraint manifold of a dyad with revolute or prismatic joints is the intersection of four linear equations with the so called Study quadric S_6^2 given by

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{1}$$

Therefore, in the analysis of the whole four link mechanisms one has to solve a system of eight linear equations and S_6^2 in the homogeneous unknowns $x_0, \dots, y_3$. Geometrically this corresponds to the intersection of eight hyperplanes with the Study quadric. To achieve mobility the eight hyperplane equations must intersect at least in a plane in order to get a curve as intersection with S_6^2, which shows that the coupler motion of a four bar mechanism with revolute and prismatic joints must be a conic section in kinematic image space. Mathematically this implies that the 8×8 coefficient matrix of the linear equation system has to have a rank deficiency of 3, in other words all 6×6 sub-determinants have to vanish identically.

For the analysis of the sub-dyads it will be convenient to use one of them in a canonical form where the axis of the first joint coincides with the z-axis of the base frame. The z-axis of the end effector frame coincides with the axis of the second joint and the foot points of the common normal of axis one and two coincide with the origins of the two frames (see Fig. 1). The second dyad consists of the remaining parts of the overall four link mechanism. This asymmetric way of dividing the mechanism gives the opportunity to solve the linear system describing the first dyad easily for four out of the eight unknowns $x_0, \dots, y_3$. Substitution of this solution into the hyperplane equations describing the second dyad yields a system of four linear equations in the four remaining unknowns. Then the aforementioned problem is reduced to a rank deficiency of 3 of the 4×4 coefficient matrix of this system. This yields the condition that all 2×2 sub-determinants have to vanish. The idea of this paper is to compute all aforementioned sub-determinants for all four link mechanisms without specifying the design. One obtains a highly redundant system of equations in the design parameters that has to be solved in order to achieve overconstrained single loop closed four link mechanisms.