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Contributions to the kinematics of pointing

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ABSTRACT

Pointing consists in orienting a line OP of a rigid body, with its point O fixed, P mobile. The operation can thus be achieved in two steps, one to provide P with the desired longitude on the sphere centered at O, of radius $\overline{OP}$, the other to provide P with the desired latitude. While this operation is simpler than a full rigid-body rotation, its kinematics is not yet fully recorded in the literature. In fact, the kinematics of pointing cannot be considered as being included in the kinematics of spherical motion, because the latter is known to form a Lie group, while pointing operations do not. This paper is an effort to shed light on the kinematics of pointing by means of the visualization of the workspace of the mechanism used for its realization. In doing this, a strategy is proposed for the planning of pointing trajectories using a three-revolute spherical wrist. The contributions proposed by the authors thus target the workspace visualization and the path-planning of three-axis wrists when used for pointing operations.

1. Background on the kinematics of rotations

The kinematics of rigid-body rotations is well documented in the literature, although most of the time the pertinent analysis is conducted based on coordinates. A coordinate-free discussion is invoked here that relies on the invariant properties of rotations, to ease the ensuing description.

First, *Euler's Theorem on rigid-body rotations is recalled* [1]: *A rigid-body motion about a point O leaves fixed a set of body points lying on a line $\mathcal{L}$ that passes through O.*

As a consequence of Euler's Theorem, a *rigid-body* rotation, or simply a *rotation*, is represented by a 3×3 *proper-orthogonal matrix* $\mathbf{Q}$—besides being orthogonal, its determinant is +1. It is known—see, for example, [2]—that the three eigenvalues of $\mathbf{Q}$ are $1,\ e^{j\phi},\ e^{-j\phi}$. The eigenvector of $\mathbf{Q}$ associated with the real eigenvalue is also real, and represented here as the unit vector $\mathbf{e}$. The real eigenvector and the two complex-conjugate eigenvalues of $\mathbf{Q}$ encapsulate the *invariant properties* of $\mathbf{Q}$, i.e., those quantities that, if scalar, remain immutable under a change of coordinate frame; if vector or matrix, these quantities undergo a *similarity transformation* [3] under such a change.

The invariants of a rotation are thus the angle ϕ through which the rigid body is rotated, and vector $\mathbf{e}$, parallel to the axis of rotation $\mathcal{L}$. These two quantities, $\mathbf{e}$ and ϕ, have been called the *natural invariants* of $\mathbf{Q}$ [4], represented henceforth by the four-dimensional array $\boldsymbol{\nu} \equiv [\mathbf{e}^T,\ \phi]^T$.

The *linear invariants* (LI) and the *Euler-Rodrigues parameters* (ERP) of the *rotation matrix*—these invariants are discussed in the literature [2,4] and [5]—are next recalled for quick reference. The LI, $\mathbf{q}$ and q_0, are defined as

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$$\mathbf{q} \equiv \mathbf{e}\sin\phi, \quad q_0 \equiv \cos\phi \tag{1}$$

with $\mathbf{q}$ extracted from $\mathbf{Q}$ via its *axial vector*, $\mathrm{vect}(\mathbf{Q})$ [2], q_0 via its trace, $\mathrm{tr}(\mathbf{Q})$, axial vector and trace also being invariant.

The Euler-Rodrigues parameters make up the other set of invariants—sometimes these invariants are referred to as the *Euler parameters*, but Chen and Gupta [6] cited historical accounts to justify their suggestion that the eponym should include Olinde Rodrigues' name as well—of the rotation in question. Represented as $\mathbf{r}$ and r_0, this set is defined as

$$\mathbf{r} \equiv \mathbf{e}\sin\left(\frac{\phi}{2}\right), \quad r_0 \equiv \cos\left(\frac{\phi}{2}\right) \tag{2}$$

with $\mathbf{r}$ extracted from the proper orthogonal square root of $\mathbf{Q}$, labeled $\sqrt{\mathbf{Q}}$, as its axial vector, r_0 via $\mathrm{tr}(\sqrt{\mathbf{Q}})$. The array of four scalars, r_0 and the three components of $\mathbf{r}$, is best known as Hamilton's *quaternion* [7].

However, to obtain the ERP from the LI of the same matrix $\mathbf{Q}$, the matrix square root is not required, as the ERP can be obtained once the LI are available, using the relations below [2]:

$$r_0 = \pm\sqrt{\frac{1+q_0}{2}}, \quad \mathbf{r} = \frac{\mathbf{q}}{2r_0}, \quad \phi \neq \pi \tag{3}$$

which work as long as $r_0 \neq 0$, i.e., as long as $\phi \neq \pi$. The special case in which $\phi = \pi$, however, can be readily handled, as it is known to yield a symmetric rotation matrix, from which $\mathbf{e}$ can be readily identified—see A.1. in Appendix.

One important feature of rigid-body rotations, that sometimes is taken for granted, is that they form a Lie group [8]; the set of rotations produced by a pointing mechanism does not. Indeed, of the 12 subgroups of the rigid-body displacement group—a subgroup of itself—listed by Hervé [8], only two are of *dimension* 2, namely, the *cylindrical*—one rotation about an axis and one translation along a direction parallel to the axis—and the *planar-translational* subgroups. In a nutshell, the dimension of a subgroup is the number of independent variables required to instantiate one member of the subgroup. In our case, the dimension can be likened to the *degree of freedom* (dof) of the mechanical system used to realize the displacements of the subgroup. As a pointing operation involves two dof, *pitch and roll* or *pan and tilt*, a "dimension" of 2 can be associated with the displacement subset in question.

What the above means is that, if the set of rotations produced by a pointing mechanism is denoted by $\mathcal{P}$, then the product $\mathbf{Q}$ of any two such rotations, say $\mathbf{Q}_1$ and $\mathbf{Q}_2$, $\mathbf{Q} = \mathbf{Q}_2\mathbf{Q}_1$, is not necessarily a member of $\mathcal{P}$[1]. This fact has important consequences, as it prevents the programming of pointing operations in the same way that rotations produced with a three-dof spherical wrist are. In robot-assisted machining, which is gaining acceptance by virtue of the low cost of robots when compared with that of CNC machining centres, a three-dof spherical wrist is available for the operation of presenting a turning cutting tool, e.g., a mill, to the workpiece. A strategy to use the capability of the three-axis spherical mechanism is needed for the two-dimensional task at hand. A sketch of such a strategy is proposed in Section 4.

2. The Rotation produced by a pointing mechanism

Although the interest in pointing mechanisms in many areas of application is significant, the kinematics of pointing is the subject of a paucity of papers. Most of the literature found on the subject of pointing pertains to the hardware, mainly electronics. In a paper on the subject, already over 20 years old, Stanišič and Duta [10] provided a list of related references, along with a novel mechanism, but of all these references, only their own work is related to the kinematics of pointing itself. More recently, work on precision systems for pointing mechanisms has been reported [11,12], that concerns itself with the compensation of "small" disturbances on the base of the pointing mechanism, but none addresses the kinematics of pointing under "large"-amplitude orientation maneuvers. The difference between "small"- and "large"-amplitude rotations cannot be overstated: the former are derived upon a linearization of the latter; this leads to matrices that, under multiplication, become commutative, as shown in A.2. in Appendix. Large-amplitude rotations are non-commutative under multiplication.

A pointing mechanism is, in general, a two-dof mechanical system, that can be constructed of either lower or higher kinematic pairs. The former, it is recalled, are characterized by a *wrapping action* of one of the two coupled links over the other [9,13]. Any other kinematic pair is of the higher type. Examples of the latter are cam and gear mechanisms. An example of pointing mechanism made of higher kinematic pairs is the *pitch-roll wrist* found in some serial robots, intended to produce pitch and roll of the robot end-effector. Such a mechanism is composed of a planetary train of bevel gears, with the morphology of an automotive differential. In this mechanism the two side gears, using the terminology of the art, are driven directly by corresponding motors, the pinions being driven by the side gears under pitching about the common axis of the side gears, and rolling about the common axis of the pinions. Moreover, one of the pinions carries the robot gripper. The focus of this paper is *pointing mechanisms made of lower kinematic pairs*. An example is the mechanism shown in Fig. 1.

Now, pointing mechanisms of the class of interest can carry, in turn, an open or a closed kinematic chain. Because of their larger mobility—a larger workspace—open chains are preferred over closed chains. A closed chain of the spherical type, to be a pointing mechanism, must be designed with two degrees of freedom, which calls for a five-link spherical linkage. An open chain needs only

[1] The order of the product indicates that the two factors are represented in the *same* coordinate frame. In the case of the Denavit-Hartenberg formalism [9], under which each rotation is represented in a specific frame associated with the rotation in question, the order is reversed.