Mechanism and Machine Theory 107 (2017) 305–323

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmachtheory

Identification and geometric characterization of Lie triple screw systems and their exponential images☆

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ARTICLE INFO

Keywords:
Screw system
Lie triple system
Constant-velocity coupling
Parallel mechanism
Type synthesis

ABSTRACT

The twist space of a plunging constant-velocity (CV) coupling with intersecting shafts consists, in all configurations, of a planar field of zero-pitch screws. Recently, we reported an important discovery about this screw system: it is closed under two consecutive Lie bracket operations, thus being referred to as a Lie triple system; taking the exponential of all its twists generates the motion manifold of the coupling. In this paper, we first give a geometric characterization of the Lie product and the Lie triple product of a generic screw system. Then, we present a systematic identification of *all* Lie triple screw systems of $\mathfrak{se}(3)$, by an approach based on both algebraic Lie group theory and descriptive screw theory. We also derive the exponential motion manifolds of the Lie triple screw systems in dual quaternion representation. Finally, several important applications of Lie triple screw systems in mechanism and machine theory are highlighted in the conclusions.

1. Introduction

Methods for inferring finite motions from infinitesimal ones are of crucial importance in kinematics of mechanisms and robotics [2,3]. For example, a one degree-of-freedom (1-DoF) finite motion is a parameterized curve in the displacement group SE(3), which may be integrated from its axode surface, a parameterized curve in the Lie algebra $\mathfrak{se}(3)$ of SE(3) [3,4]. From mechanism synthesis point of view, it is important to consider multi-DoF finite motions or *parameterized submanifolds* of SE(3) that may be generated by the end-effector of an open-chain or closed-chain mechanism [5–9]. Several researchers characterize the end-effector motion set of mechanisms by the more general notion of algebraic varieties [10,11]. In practice, we may always assume that such varieties are submanifolds by restricting them to an open subset about the identity element of SE(3).

Inferring finite end-effector motions of mechanisms from their *twist space*[1] has been extensively studied in the literature, in particular in type synthesis of parallel mechanisms [13,5–7,14,8,15,9,16,17,12,1]. The methods of inference involved in such works are mainly heuristic and developed on a case-by-case basis. Hunt pointed out that the full-cycle mobility of a kinematic chain may be inferred from its twist space if the latter remains invariant for all configurations [2]. The inferred finite motions are simply the connected subgroups of SE(3), with their twist spaces being the corresponding Lie subalgebras of $\mathfrak{se}(3)$ [18,5,19]. For this reason, Lie subalgebras of $\mathfrak{se}(3)$ are sometimes referred to as *invariant screw system* (ISS) [12]. More generally, when the twist space of a mechanism remains invariant *up to a rigid displacement* under arbitrary finite motions away from singularities, thus having a

☆ A preliminary version of this paper was presented at the 14th IFTOMM World Congress, Taiwan, Oct. 25–30, 2015 [1].

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[1] The twist space of a mechanism is the vector space of all possible end-effector twists at a given configuration. In other words, it is a vector subspace of the Lie algebra $\mathfrak{se}(3)$ of the special Euclidean group SE(3).

http://dx.doi.org/10.1016/j.mechmachtheory.2016.09.020
Received 23 November 2015; Received in revised form 21 September 2016; Accepted 22 September 2016

Nomenclature

$\mathcal{R}$	revolute joint
$\mathcal{S}$	spherical-joint-equivalent subchain
$\mathcal{E}$	planar-joint-equivalent subchain
	zero-pitch screw
	infinite-pitch screw
	finite-non-zero-pitch screw
SE(3)	the special Euclidean group
$\mathfrak{se}(3)$	the Lie algebra of SE(3)
$\boldsymbol{\xi}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$	twists or screws
S	generic screw system with unspecified type
$S_{i,j}$	the jth special i-system (see [2, Ch. 12])
$S^0_{i,j}$	jth special i-system whose finite-pitch screws all have zero pitch
$L, L_{i,j}, L^0_{i,j}$	Lie product of S, $S_{i,j}$ and $S^0_{i,j}$ respectively: $L = [S, S]$
$T, T_{i,j}, T^0_{i,j}$	triple product of S, $S_{i,j}$ and $S^0_{i,j}$ respectively: $T = [[S, S], S]$

constant *class* [20], the mechanism is said to have a *persistent screw system* (PSS) of the end-effector (see Fig. 1 for examples) [12]. The notion of PSS is important as it allows us to expand the theoretical framework under which the inference of finite motion properties from instantaneous ones is guaranteed. The exhaustive derivation and classification of all *serial* chains with an m-dimensional (m-D) PSS of the end-effector is complete up to $m \le 4$ [21–23].

However, not all persistent motions are generated by serial chains. In a seminal paper on the analysis and synthesis of CV couplings with parallel-kinematics structure [24], Hunt identified the twist space of a CV coupling with intersecting shafts to be either the fourth special three-system (with zero-pitch screws), denoted $S^0_{3,4}$, or the first special two-system (with zero-pitch screws), denoted $S^0_{2,1}$, depending on whether plunging is allowed or not (see Fig. 2 and also [2, Ch. 13.5]). Focusing, without loss of generality, on the plunging CV coupling, CV transmission is provided for any input and articulation angles if the twist space always remains a $S^0_{3,4}$ system, with all its screws lying on the bisecting plane [24]. The persistence property of the coupling is enforced by a parallel-kinematics architecture with prescribed types of connecting chains with five (in some cases four) DoF. All of the latter respect mirror symmetry about the bisecting plane (see Fig. 3 and also [24,15]). The twist space of a CV coupling with intersecting shafts, although not fixed at a particular location, remains congruent to itself under arbitrary configuration changes, thus being persistent. It is briefly stated in [24] and later proved in [21] that there exists no serial chain whose twist space remains $S^0_{3,4}$ for all its configurations. The serial 5-DoF connecting chains in a plunging CV coupling are not persistent either. For example, the twist space of the 5$\mathcal{R}$ chain shown in Fig. 3(b) has a reciprocal screw with zero pitch at a mirror symmetric configuration [24]; however, the chain may also have configurations where the reciprocal screw has infinite pitch [25]. This shows that the persistent behavior of $S^0_{3,4}$ in a CV coupling is not the result of the in-parallel connection of higher-dimensional-PSS serial generators.

Bonev et al. [26] studied CV-like generators of $S^0_{3,4}$ from a finite-motion perspective, by the so-called tilt-and-torsion orientation parameters. They pointed out that the end-effector motion of a 3 − $\mathcal{RSR}$ reflected tripod, which is a parallel mechanism equivalent to a plunging CV coupling (see [2,15, pp. 397]), has zero torsion for all its configurations. In other words, the reflected tripod has a zero-torsion motion type[2] [27]. It is further shown in [27] that the motion type of a plunging CV coupling is exactly the image $\exp(S^0_{3,4})$ of $S^0_{3,4}$ under the exponential map exp.[3] The fact that its tangent spaces are all copies of $S^0_{3,4}$ can be readily verified by the half-angle property [28,29].

Through a series of recent studies, we discovered that the many geometric properties of $\exp(S^0_{3,4})$ are best understood from a symmetric space viewpoint [28,29]. A symmetric space is a smooth manifold which has an inversion symmetry about every point [30]. The inversion symmetry condition for a screw system S is [28,29]:

$$\forall\ \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S: \quad e^{\boldsymbol{\xi}_1} e^{\boldsymbol{\xi}_2} e^{\boldsymbol{\xi}_1} \in \exp(S) \tag{1}$$

Both $S^0_{2,1}$ and $S^0_{3,4}$ satisfy condition (1), turning $\exp(S^0_{2,1})$ and $\exp(S^0_{3,4})$ into symmetric spaces. We showed in [29] that the inversion symmetry of S leads to the mirror symmetric arrangement of joint axes in a CV coupling. The necessary and sufficient condition for a screw system S to satisfy (1) is being closed under two consecutive *Lie bracket* or *commutator* [19] operations, namely [30]

$$\forall\ \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in S: \quad [[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2], \boldsymbol{\xi}_3] \in S \tag{2}$$

The double commutator operation $[[\cdot,\cdot], \cdot]$ is sometimes referred to as a *Lie triple product* [31]. A screw system S that satisfies (2) is

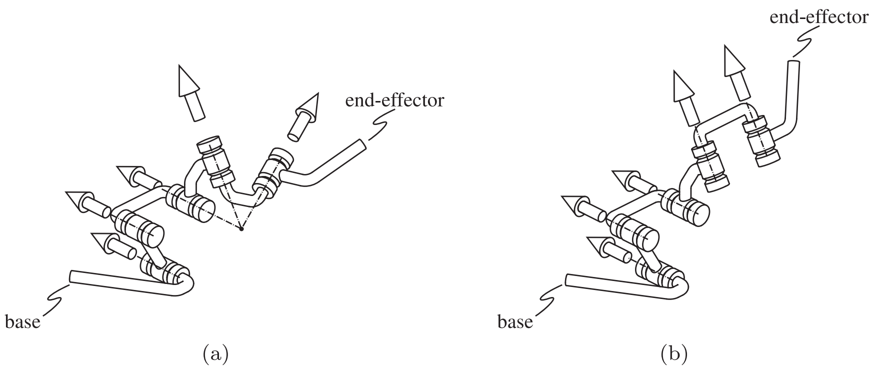

Fig. 1. Examples of PSS serial generators [12]: (a) a $\mathcal{ES}$-type serial chain; (b) $\mathcal{EE}$-type serial chain.