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Closed-form position analysis of variable geometry trusses

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ABSTRACT

Variable geometry trusses are composed, in general, of unit cells which can be modeled as bars connected by spherical joints. Under mild conditions, it has been shown that the only feasible cells are topologically equivalent to bipyramids. Unfortunately, using standard formulations, the closed-form position analysis of bipyramids is not a trivial task. Actually, it has only been achieved for bipyramids with up to 7 vertices, whose closure polynomial has been shown to be of order 24. In this paper, using a distance-based formulation and a kinematic inversion for fans of tetrahedra, the problem is solved for bipyramids with up to 11 vertices, whose closure polynomial is of degree 896. No other position analysis problem leading to such a high-order closure polynomial has been previously solved.

1. Introduction

A truss is a structure where each element, typically a bar, only supports tension or compression forces because it is connected to other bars through what are assumed to be multiple spherical joints [1], although in some cases the joints may be separated in the actual construction [2,3]. While rigid trusses have been widely used in construction, passive mobile trusses are commonly used, for instance, as shock absorbers. The advent of automation opened the possibility to build active trusses, i.e., trusses which can actively vary their geometry as needed [4]. The motion of such devices is commonly achieved by having actuated bars with variable length. Actually, the number of such bars gives the number of degrees of freedom of the device. Arbitrary variable geometry trusses may be defined but, to facilitate their design, analysis, construction, and control, they are typically built with repetitions of a given unit cell. A mechanism composed of several unit cells can have a large workspace, like a serial robot, but at the same time the high stiffness of a parallel one. This is why variable geometry trusses are sometimes considered a generalization of the serial/parallel robots [5]. Due to their exceptional stiffness to weight ratio, their structural simplicity, and their shape versatility, variable geometry trusses have a myriad of potential applications including robot arms [6], hyper-redundant manipulators [7], flight simulators [8], payload vibration reduction systems [9], tools to manipulate large payloads [10], morphing wings [11], space devices [12] or civil engineering structures [13].

The design of novel variable geometry trusses rely on having a complete kinematics characterization of their constituent cells. The larger the set of cells whose position analysis is solved, i.e., the set of cells whose assembly modes can be obtained solely from their edge lengths, the larger the options available to the designer. The motion capabilities and the ways to actuate a cell increase with its number of bars, but this also complicates its position analysis.

It was shown in [5] that, under mild conditions, the only feasible cells in variable geometry trusses are topologically equivalent to bipyramids, i.e, polyhedral resulting from joining two n-gonal pyramids at their bases (see Table 1). Note, thought, that unit cells must not be decomposable in combinations of simpler unit cells. Thus, the bipyramid of order 1 in Table 1, i.e., the hexahedron,

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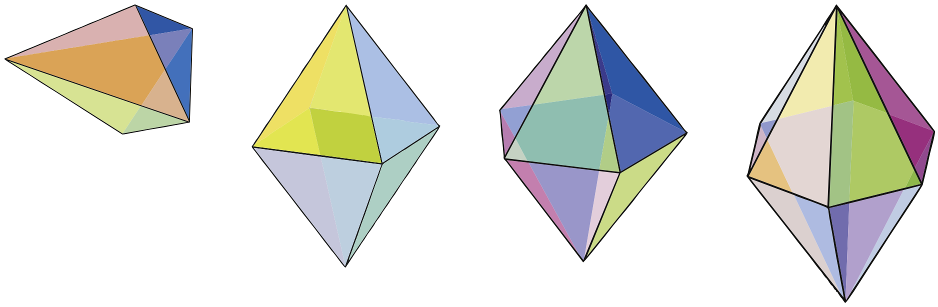
Table 1
The cells traditionally considered as the building blocks of variable geometry trusses.



Truss	Hexahedron	Octahedron	Decahedron	Dodecahedron
Order (N)	1	2	3	4
Vertices ($N+4$)	5	6	7	8
Faces ($2N+4$)	6	8	10	12
Edges ($3N+6$)	9	12	15	18
Spherical joints ($5N+8$)	13	18	23	28
Quadratic equations ($3N+3$)	6	9	12	15
Polynomial degree ($N2^N$)	2	8	24	64
References	[14]	[15,3]	[16]	[17]

cannot be properly considered a unit cell since it can be separated into two tetrahedra. We include it in the discussion because it will be the basic element used to derive the univariate closure polynomial of higher order bipyramids.

The position analysis of bipyramids can be addressed using numerical techniques such as those based on continuation [18,19], or interval analysis [20,21]. However, closed-form solutions in the form of univariate closure polynomials are, in general, preferable because they offer more information on the problem, such as an upper bound on the number of assembly modes, or the possible structure of the singularity set. Closed-form solutions are typically obtained using elimination techniques [22] where a univariate closure polynomial is obtained applying algebraic manipulations, sometimes driven by intuition, on a set of equations resulting from an algebraic formulation of the problem. Thus, the first step to solve the problem is to obtain a good formulation.

A straightforward formulation of the position analysis of bipyramids results from assigning coordinates to three vertices defining a face, and leaving all other vertices' coordinates as variables. These variable coordinates are obviously constrained by the edge lengths which translate into quadratic equations relating them. The result is a large sparse system of equations from which it is not easy to obtain a univariate resultant without introducing extraneous factors [23]. As an alternative, ad hoc approaches for particular cells permit obtaining more compact formulations by introducing position and angular variables. In this case, the former variables are usually put in terms of the latter, and the tangent half-angle substitution is used to obtain a reduced system of algebraic expressions.

Unfortunately, even with compact formulations, the elimination process gets rapidly involved as the order of the bipyramid increases. Therefore, up to now, only closed-form solutions for the octahedral [15,3] and the decahedral bipyramids [16] are available. For these two cases, it has been shown that the univariate closure polynomials are of degree 8 and 24, respectively. The following bipyramid in complexity, the dodecahedral bipyramid of order 4, has only been analyzed using numerical techniques [17].

In this paper, the position analysis problem is solved for bipyramids up to order 7 using a coordinate-free distance-based formulation and a kinematic inversion for fans of tetrahedra (a sequence of tetrahedra sharing a common edge). The presented method starts by finding the distance between the end-vertices of a fan of k tetrahedra, i.e., its vertices of degree 3. Then, a univariate closure polynomial for a $(k-1)$-order bipyramid is obtained using a kinematic inversion. The procedure is surprisingly simple, and, from our point of view, its ability to obtain a closure polynomial without using any kind of variable elimination is remarkable.

This paper is organized as follows. Section 2 shows how to obtain the distance between the end-vertices of a tetrahedral fan with an arbitrary number of tetrahedra. Then, Section 3 uses this result to derive a univariate closure polynomials for bipyramids using a kinematic inversion. Section 4 solves the position analysis of bipyramids up to order 7, and, finally, Section 5 summarizes and discusses the main contributions of the proposed approach.

2. The distance geometry of fans of tetrahedra

The valid distances between a set of points can be characterized using the theory of Cayley–Menger determinants [24,25]. The Cayley–Menger bi-determinant of two sets of points, $P_{i_1},\ldots,P_{i_n}$ and $P_{j_1},\ldots,P_{j_n}$, is defined[1] as

[1] An alternative common definition (see for instance [24]) includes a constant factor, which is dropped here to simplify the formulation.