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Sparse polynomial chaos expansions of frequency response functions using stochastic frequency transformation

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A B S T R A C T

Frequency response functions (FRFs) are important for assessing the behavior of stochastic linear dynamic systems. For large systems, their evaluations are time-consuming even for a single simulation. In such cases, uncertainty quantification by crude Monte-Carlo simulation is not feasible. In this paper, we propose the use of sparse adaptive polynomial chaos expansions (PCE) as a surrogate of the full model. To overcome known limitations of PCE when applied to FRF simulation, we propose a frequency transformation strategy that maximizes the similarity between FRFs prior to the calculation of the PCE surrogate. This strategy results in lower-order PCEs for each frequency. Principal component analysis is then employed to reduce the number of random outputs. The proposed approach is applied to two case studies: a simple 2-DOF system and a 6-DOF system with 16 random inputs. The accuracy assessment of the results indicates that the proposed approach can predict single FRFs accurately. Besides, it is shown that the first two moments of the FRFs obtained by the PCE converge to the reference results faster than with the Monte-Carlo (MC) methods.

1. Introduction

Interest towards working with large engineering systems is increasing recently, but long simulation time is one of the main limiting factors. Although the development of the computational power of modern computers has been very fast in recent years, increasing model complexity, more precise description of model properties and more detailed representation of the system geometry still result in considerable execution time and memory usage. Model reduction [1,2], efficient simulation [3–5] and parallel simulation methods [6,7] are different strategies to address this issue.

Consequently, uncertainty propagation in these systems cannot be carried out by classical approaches such as crude Monte-Carlo (MC) simulation. More advanced methods such as stochastic model reduction [8] or surrogate modeling [9] are required to replace the computationally expensive model with an approximation that can reproduce the essential features faster. Of interest here are surrogate models. They can be created intrusively or non-intrusively. In intrusive approaches, the equations of the system are modified such that one explicit function relates the stochastic properties of the system responses to the random inputs. The perturbation method [10] is a classical tool used for this purpose but it is only accurate when the random inputs have small coefficients of variation (COV). An alternative method is intrusive polynomial chaos expansion [11]. It was first introduced for Gaussian input random variables [12] and then extended to the other types of random variables leading to generalized polynomial chaos [13,14].

In non-intrusive approaches, already existing deterministic codes are evaluated at several sample points selected over the parameter space. This selection depends on the methods employed to build the surrogate model, namely regression [15,16] or projection methods [17,18]. Kriging [19,20] and non-intrusive PCE [21] or combination thereof [22,23] are examples of the non-intrusive approaches. The major drawback of PCE methods, both intrusive and non-intrusive, is the large number of unknown coefficients in problems with large parameter spaces, which is referred to as the curse of dimensionality [24]. Sparse [25] and adaptive sparse [26] polynomial chaos expansions have been developed to dramatically reduce the computational cost in this scenario.

To propagate and quantify the uncertainty in a Quantity of Interest (QoI) of a system, its response should be monitored all over the parameter space. This response could be calculated in time, frequency or modal domain. For dynamic systems, the frequency response is important because it provides information over a frequency range with a clear physical interpretation. This is the main reason of the recent focus on frequency response functions (FRF) for uncertainty quantification of dynamic systems and their surrogates [19,27–30].

Several attempts have been made to find a surrogate model for the FRF by using modal properties or random eigenvalue problems. Pichler

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et al. [31] proposed a mode-based meta-model for the frequency response functions of stochastic structural systems. Yu et al. [32] used Hermite polynomials to solve the random eigenvalue problem and then employed modal assurance criteria (MAC) to detect the phenomenon of modal intermixing. Manan and Cooper [33] used non-intrusive polynomial expansions to find the modal properties of a system and predict the bounds for stochastic FRFs. They implemented the method on models with one or two parameters and COV ≤2%.

Very few and recent papers addressed the direct implementation of PCE on the frequency responses of systems. Kundu and Adhikari [34] proposed to obtain the frequency response of a stochastic system by projecting the response on a reduced subspace of eigenvectors of a set of complex, frequency-adaptive, rational stochastic weighting functions.

Pagnacco et al. [35] investigated the use of polynomial chaos expansions for modeling multimodal dynamic systems using the intrusive approach by studying a single degree of freedom (DOF) system. They showed that the direct use of the polynomial chaos results in some spurious peaks and proposed to use multi-element PCE to model the stochastic frequency response but, to the knowledge of the authors, they did not publish anything on more complex systems yet. Jacquelin et al. [36] studied a 2-DOF system to investigate the possibility of direct implementation of PCE for the moments of the FRFs and they also reported the problem of spurious peaks. They showed that the PCE converges slowly on the resonance parts. They accelerate the convergence of the first two statistical moments by using Aitken's method and its generalizations [37].

In general, there are two main difficulties to make the PCE surrogate model directly for the FRFs: (i) their non-smooth behavior over the frequency axis due to abrupt changes of the amplitude that occur close to the resonance frequencies. At such frequencies, the amplitudes are driven by damping [38]. In [39], Adhikari and Pascal investigated the effect of damping in the dynamic response of stochastic systems and explain why making surrogate models in the areas close to the resonance frequencies is very challenging. (ii) the frequency shift of the eigenfrequencies due to uncertainties in the parameters. This results in very high-order PCEs even for the FRFs obtained from cases with 1 or 2 DOFs. The main contribution of this work is to propose a method that can solve both problems.

The proposed approach consists of two steps. First, the FRFs are transformed via a stochastic frequency transformation such that their associated eigenfrequencies are aligned in the transformed frequency axis, called *scaled frequency*. Then, PCE is performed on the *scaled frequency* axis.

The advantage of this procedure is the fact that after the transformation, the behavior of the FRFs at each *scaled frequency* is smooth enough to be surrogated with low-order PCEs. However, since PCE is made for each *scaled frequency*, this approach results in a very large number of random outputs. To solve this issue, an efficient version of principal component analysis is employed. Moreover, the problem of the curse of dimensionality is resolved here by means of adaptive sparse PCEs.

The outline of the paper is as follows. In Section 2, the required equations for deriving the FRFs of a system are presented. In Section 3, all appropriate mathematics for approximating a model by polynomial chaos expansion are presented. The main challenges for building PCEs for FRFs are elaborated and the proposed solutions are presented. In Section 4, the method is applied to two case studies, a simple case and a case with a relatively large number of input parameters.

2. Frequency response function (FRF)

Consider the spatially-discretized governing second-order equation of motion of a structure as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{V}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f}(t) \tag{1}$$

where for an n-DOF system with n_u system inputs and n_y system outputs, $\boldsymbol{q}(t) \in \mathbb{R}^n$ is the displacement vector, $\boldsymbol{f}(t)$ is the external load vector which is governed by a Boolean transformation of stimuli vector $\boldsymbol{f}(t) = \boldsymbol{P}_u\boldsymbol{u}(t)$; with $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$. Real positive-definite symmetric matrices $\boldsymbol{M}, \boldsymbol{V}, \boldsymbol{K} \in \mathbb{R}^{n\times n}$ are mass, damping and stiffness matrices, respectively. The state-space realization of the equation of motion in Eq. (1) can be written as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{2}$$

where $\boldsymbol{A} \in \mathbb{C}^{2n\times 2n}$, $\boldsymbol{B} \in \mathbb{C}^{2n\times n_u}$, $\boldsymbol{C} \in \mathbb{C}^{n_y\times 2n}$, and $\boldsymbol{D} \in \mathbb{C}^{n_y\times n_u}$. $\boldsymbol{x}^T(t) = [\boldsymbol{q}(t)^T, \dot{\boldsymbol{q}}^T(t)] \in \mathbb{R}^{2n}$ is the state vector, and $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ is the system output. $\boldsymbol{A}$ and $\boldsymbol{B}$ are related to mass, damping and stiffness as follows

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{V} & -\boldsymbol{M}^{-1}\boldsymbol{K} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{P}_u \end{bmatrix}. \tag{3}$$

The output matrix $\boldsymbol{C}$, which has application dependent elements, linearly maps the states to the output $\boldsymbol{y}$ and $\boldsymbol{D}$ is the associated direct throughput matrix. The frequency response of the model (2) can be written as

$$\mathcal{H}(j\omega) = \boldsymbol{C}(j\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}, \tag{4}$$

where $\mathcal{H} = [\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{n_u\times n_y}]^{\mathrm{T}} \in \mathbb{C}^{(n_y\times n_u)\times 1}, \forall\ \omega$ and $j = \sqrt{-1}$. $(\bullet)^{\mathrm{T}}$ stands for the transpose of the matrix. It should be mentioned that the eigenvalues of $\boldsymbol{A}$ are the poles of the system. They are complex and their imaginary parts can be approximated as the frequencies, in rad/s, at which the maximum amplitude occurs.

3. Methodology

This section first, briefly reviews polynomial chaos expansion for real-valued responses. Then, the method of stochastic frequency transformation is explained in conjunction with the proposed method as well as its application to the complex-valued FRF responses.

3.1. Polynomial chaos expansions

Let $\mathcal{M}$ be a computational model with M-dimensional random inputs $\boldsymbol{X}=\{X_1, X_2, \ldots, X_M\}^{\mathrm{T}}$ and a scalar output Y. Further, let us denote the joint probability distribution function (PDF) of the random inputs by $f_{\boldsymbol{X}}(\boldsymbol{x})$ defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that the system response $Y = \mathcal{M}(\boldsymbol{X})$ is a second-order random variable, i.e. $\mathbb{E}[Y^2] < +\infty$ and therefore it belongs to the Hilbert space $\mathcal{H} = \mathcal{L}^2_{f_X}(\mathbb{R}^M, \mathbb{R})$ of f_X-square integrable functions of $\boldsymbol{X}$ with respect to the inner product:

$$\mathbb{E}[\psi(\boldsymbol{X})\phi(\boldsymbol{X})] = \int_{\mathcal{D}_X} \psi(\boldsymbol{x})\phi(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{5}$$

where $\mathcal{D}_X$ is the support of $\boldsymbol{X}$. Further assume that the input variables are independent, i.e. $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{M} f_{X_i}(x_i)$. Then the generalized polynomial chaos representation of Y reads [13]:

$$Y = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^M} \tilde{u}_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \tag{6}$$

in which $\tilde{u}_{\boldsymbol{\alpha}}$ is a set of unknown deterministic coefficients, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ is a multi-index set which indicates the polynomial degree of $\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$ in each of the M input variables. $\psi_{\boldsymbol{\alpha}}$ s are multivariate orthonormal polynomials with respect to the joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$, i.e. :

$$\mathbb{E}\left[\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})\psi_{\boldsymbol{\beta}}(\boldsymbol{X})\right] = \int_{\mathcal{D}_X} \psi_{\boldsymbol{\alpha}}(\boldsymbol{x})\psi_{\boldsymbol{\beta}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \tag{7}$$

where $\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ is the Kronecker delta. Since the input variables are assumed to be independent, these multivariate polynomials can be constructed by a tensorization of univariate orthonormal polynomials with respect to the