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Simulation of higher-order stochastic processes by spectral representation

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ABSTRACT

The Spectral Representation Method is generalized for simulation of asymmetrically nonlinear (skewed higher-order) stochastic processes. This is achieved by deriving new orthogonal increments for the spectral process in the Cramér spectral representation that include wave interactions and satisfy third-order orthogonality properties. These orthogonal increments are derived by introducing two new quantities – the pure power spectrum and the partial bicoherence – that decouple the contributions of single waves and wave interactions in the Fourier-type expansion of a stochastic process. The further extension to fourth and higher-order processes is discussed. Several mathematical examples demonstrate the capabilities of the proposed methodology to generate general third-order stochastic processes. The method is then applied to the generation of turbulent wind velocities characterized from Large Eddy Simulations of the atmospheric boundary layer.

1. Introduction

Stochastic process theory has wide-ranging applications in engineering mechanics from characterizing and synthesizing heterogeneous materials to the dynamics of ocean waves, wind loads, and earthquake accelerations. Computational analysis of these stochastic systems requires the generation of realistic sample functions of stochastic processes. These simulations are typically applied in the context of a Monte Carlo type analysis wherein a large number of sample functions are simulated, the system of governing equations are solved for each simulation, and the results are evaluated statistically. Analogously, these simulations may occur in the context of a stochastic collocation, quasi-Monte Carlo, variance reduced Monte Carlo, or some other simulation-based probabilistic method where the intention is to quantify the response probabilistically (e.g. to assess variability or reliability).

The crux of these simulation-based approaches is the ability to accurately generate realizations of the stochastic processes that possess the desired probability law to an acceptable degree. This problem has given rise to many different simulation methodologies over the past 40 years. We will focus in this work on those methodologies constructed from stochastic expansions of general form:

$$A(x,\omega) \approx \widehat{A}(x,\omega) = \sum_{i=1}^{n} C_i(\omega)\theta_i(x), \quad x \in D, \omega \in \Omega \tag{1}$$

where $\{C_i(\omega)\}$ are a set of independent random variables on the probability space (Ω, F, P) and $\{\theta_i(x)\}$ are deterministic basis functions. Most notable for our purposes is that, to date, every such stochastic expansion has been derived from second-order properties of the process (specifically from the covariance function). The most commonly used of these second-order expansions are the Karhunen–Loève (K–L) expansion [1,2], the spectral representation method (SRM) [3–5], and optimal linear expansion (OLE) [6]. Each of these methods expands the process according to Eq. (1) by identifying the random variables $\{C_i(\omega)\}$ and the basis functions $\{\theta_i(x)\}$ such that

$$C(x_1,x_2) = E[A(x_1)A(x_2)] \approx E[\widehat{A}(x_1)\widehat{A}(x_2)] \tag{2}$$

In the case of the K–L expansion, $\{\theta_i(x)\}$ are the eigenfunctions of $C(x_1, x_2)$ and $\{C_i(\omega)\}$ are zero-mean and unit standard deviation random variables. Similarly, for the SRM, $\{\theta_i(x)\}$ are harmonic functions and the $\{C_i(\omega)\}$ can be derived from the power spectral density function (Fourier transform of $C(x_1, x_2)$).

The primary drawback of these methodologies is their inherently second-order nature - they can only represent the process accurately up to its covariance. This second-order limitation equates to the assumption that the stochastic process arises as the output of a linear system operating on some random input. This is clearly not true for many real stochastic processes generated from strongly nonlinear systems such as turbulent flow governed by the Navier–Stokes equations, seismic wave propagation in nonlinear soils, or through systems approximated using say, N-th order Volterra series [7,8]. These nonlinear systems result in complex nonlinear dependencies and wave interactions that are elaborated in the subsequent sections. For this reason, such processes have been commonly referred to, and are referred to herein as nonlinear processes.

Expansions with the general form in Eq. (1) are shown to be asymptotically Gaussian when $\{C_i(\omega)\}$ are independent by the Central limit theorem [9]. Efforts toward improving these models have focused almost exclusively on nonlinear transformations of Eq. (1) – often referred to as translation processes [10]. These nonlinear transformations come in a variety of forms that map the process to match specific properties of the process such as moments of a specified order using e.g.

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Hermite polynomials [11,12] – while matching the covariance function as closely as possible. One class of nonlinear transformations of particular interest are those capable of matching the marginal non-Gaussian PDF of the process and the covariance function through either the explicit CDF-based mapping [10,13]

$$Y(x) = F^{-1}\{\Phi[A(x)]\} \tag{3}$$

where $A(x)$ is a Gaussian random process, $\Phi(\cdot)$ is the standard normal CDF and $F^{-1}(\cdot)$ is the inverse non-Gaussian marginal CDF, or through an approximate polynomial chaos expansion [14]. Recent developments have enabled the efficient and widespread use of these CDF-based translation models for simulation of scalar [15] and vector [16], stationary processes and nonstationary processes [17,18] using either the SRM or K–L expansion.

Another class of methods, first proposed by Phoon et al. [19,20] aims to expand a second-order non-Gaussian process directly using the K–L expansion by iteratively identifying the non-Gaussian distribution of the K–L random variables. Unlike the nonlinear translation-based approaches, however, these methods are not able to match the marginal non-Gaussian PDF of the process exactly.

The aforementioned methodologies represent notable achievements in producing high-accuracy stochastic process simulations. Their common shortcoming is that they all remain second-order, derived only from the covariance of the process and marginal densities. In this work, we derive a novel direct higher-order stochastic expansion that generalizes the popular Spectral Representation Method. Specifically, we derive orthogonal increments for the classical Cramér spectral representation that satisfy higher-order orthogonality conditions elucidated by Rosenblatt and Van Ness [21]. Applying these orthogonal increments to a stochastic expansion model in the form of the spectral representation provides a means of generating sample functions of stochastic processes with prescribed 3-point correlations (equivalently bispectra). Some discussion of generalization to processes with prescribed n-point correlations is provided. We explore the implications of the proposed method through several mathematical examples and apply it to the generation of a wind velocity process characterized from computational fluid dynamics (CFD).

Prior to deriving the new methodology, we provide a review of higher-order properties of stochastic processes and their importance in mechanics in the following section.

2. Higher-order properties of stochastic processes

The higher-order properties are basic descriptors of stochastic processes resulting from nonlinear systems (here, nonlinear mechanics) [22]. Integrating these higher-order properties can be critical to the accurate characterization and modeling of numerous physical systems. Effective material properties for random heterogeneous materials (e.g. electrical or thermal conductivity, permeability in porous media), for example, are often governed by complex n-point correlations, lineal-path functions, etc. that are not captured by simply matching the covariance of the process [23]. This has motivated methodologies for random material morphology generation using inverse optimization approaches wherein a specific sample function is stochastically optimized to match these higher-order properties [24]. These methods, however, involve solving a computationally expensive stochastic optimization problem for each generated sample function with no guarantee that the properties of the process will match with sufficient accuracy.

Additional applications in mechanics where higher-order properties have been shown to be important include nonlinear dynamics [25,26] and various areas of fluid mechanics such as wave interactions [27,28] and turbulence [29,30]. Moreover, the importance of higher-order properties in stochastic processes goes far beyond the applications in mechanics to signal processing where they are widely used [31,32,26], analysis of astronomical data [33], and plasma physics [34]. However, in all of the previous examples (materials modeling excepted), the focus has been entirely on characterization and interpretation of higher-order properties and there have been no efforts to integrate these important properties in the modeling of physical systems. This, we believe is driven by two factors. 1. Characterization of higher-order properties presents numerous technical challenges that can only be overcome by access to very large data sets. 2. No mathematical construction currently exists through which to integrate these properties. We aim to address the second challenge specifically and believe that the first challenge is presently being addressed through massive data collection efforts that are growing increasingly common in numerous fields (e.g. materials characterization and wind engineering).

Having motivated their importance, we now briefly review the specific mathematical properties of interest here.

2.1. Cumulant and moment functions

Given a real random vector $X = \{x_1, x_2, \ldots, x_n\}$, the joint moments of order $r = k_1 + k_2 + \cdots + k_n$ are defined by [35,31]:

$$m_{k_1,k_2,\ldots,k_n} \equiv E[x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}] = (-i)^r \left. \frac{\partial^r \Phi(\omega_1, \omega_2, \ldots, \omega_n)}{\partial\omega_1^{k_1} \omega_2^{k_2} \cdots \partial\omega_n^{k_n}} \right|_{\omega_1=\omega_2=\cdots=\omega_n=0} \tag{4}$$

and the joint cumulants of order r are defined by:

$$c_{k_1,k_2,\ldots,k_n} \equiv (-i)^r \left. \frac{\partial^r \ln \Phi(\omega_1, \omega_2, \ldots, \omega_n)}{\partial\omega_1^{k_1} \omega_2^{k_2} \cdots \partial\omega_n^{k_n}} \right|_{\omega_1=\omega_2=\cdots=\omega_n=0} \tag{5}$$

where

$$\Phi(\omega_1, \omega_2, \ldots, \omega_n) \equiv E[e^{i(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n)}] \tag{6}$$

is the joint characteristic function. In general, the cumulants can be expressed in terms of the moments through the following relationships [31]:

$$c_{k_1,k_2,\ldots,k_n} = \sum (-1)^{p-1}(p-1)!E\left[\prod_{i\in s_1} x_i\right]E\left[\prod_{i\in s_2} x_i\right]\cdots E\left[\prod_{i\in s_p} x_i\right] \tag{7}$$

where the summation extends over all groups $\{s_1, s_2, \ldots, s_p\}$, $p = 1, 2, \ldots, n$ of the integers $k_1, k_2, \ldots, k_n$. For example, some third order cumulants are given by

$$\begin{aligned} c[x_1, x_2, x_3] &= E[x_1x_2x_3] - E[x_1]E[x_2x_3] - E[x_2]E[x_1x_3] - E[x_3]E[x_1x_2] \\ &\quad + 2E[x_1]E[x_2]E[x_3] \\ c[x_1, x_2^2] &= c[x_1, x_2, x_2] = E[x_1x_2^2] - E[x_1]E[x_2^2] - 2E[x_2]E[x_1x_2] \\ &\quad + 2E[x_1]E[x_2]^2 \\ c[x_1^3] &= c[x_1, x_1, x_1] = E[x_1^3] - 3E[x_1]E[x_1^2] + 2E[x_1]^3 \end{aligned} \tag{8}$$

Note that when X is a jointly Gaussian random vector, all cumulants of order $n > 2$ are identically zero. In this sense, the cumulants of order higher than $n = 2$ provide measures of non-Gaussianity.

Next, let $f(t)$ denote a real stationary stochastic process such that the moment and cumulant functions can be denoted by:

$$\begin{aligned} m[f(t), f(t+\tau_1), \ldots, f(t+\tau_{n-1})] &\equiv E[f(t)f(t+\tau_1)\cdots f(t+\tau_n)] \\ &= m_n^f(\tau_1, \tau_2, \cdots\tau_{n-1}) \\ c[f(t), f(t+\tau_1), \ldots, f(t+\tau_{n-1})] &\equiv c_n^f(\tau_1, \tau_2, \cdots\tau_{n-1}) \end{aligned} \tag{9}$$

Combining Eqs. (7) and (9), the cumulant functions can be expressed in terms of the moment functions as:

$$\begin{aligned} c_1^f &= m_1^f \\ c_2^f(\tau) &= m_2^f(\tau) - m_1^f \\ c_3^f(\tau_1, \tau_2) &= m_3^f(\tau_1, \tau_2) - m_1^f[m_2^f(\tau_1) + m_2^f(\tau_2) + m_2^f(\tau_2 - \tau_1)] + 2(m_1^f)^3 \\ c_4^f(\tau_1, \tau_2, \tau_3) &= m_4^f(\tau_1, \tau_2, \tau_3) - m_2^f(\tau_1)m_2^f(\tau_2 - \tau_3) \\ &\quad - m_2^f(\tau_2)m_2^f(\tau_3 - \tau_1) - m_2^f(\tau_3)m_2^f(\tau_2 - \tau_1) \\ &\quad - m_1^f[m_3^f(\tau_2 - \tau_1, \tau_3 - \tau_1) + m_3^f(\tau_2, \tau_3) + m_3^f(\tau_3, \tau_1) \\ &\quad + m_3^f(\tau_1, \tau_2)] + (m_1^f)^2[m_2^f(\tau_1) + m_2^f(\tau_2) + m_2^f(\tau_3) \\ &\quad + m_2^f(\tau_3 - \tau_1) + m_2^f(\tau_3 - \tau_2) + m_2^f(\tau_2 - \tau_1)] + 6(m_1^f)^4 \\ &\quad \vdots \end{aligned} \tag{10}$$

Notice that when $f(t)$ is a zero mean process ($m_1^f = 0$), the moments and cumulants up to order 3 are identical.