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A direct simulation method and lower-bound estimation for a class of gamma random fields with applications in modelling material properties

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ABSTRACT

Existing approaches for generating non-Gaussian random fields typically utilize translation process theory that applies a memoryless nonlinear transformation to an underlying Gaussian random field. In the current study, a direct non-translation approach is proposed to generate random fields with a marginal gamma distribution. The proposed approach is based on the additive reproductive property of the gamma distribution; it results in a conceptually simple algorithm that is straightforward to implement in Monte-Carlo simulations. It is demonstrated that an arbitrary marginal gamma distribution is achievable. The resulting auto-correlation functions are non-negative and decreasing functions with a prescribed scale of fluctuation. These characteristics make the proposed non-translation approach suitable for modelling the spatial variability in material properties. The engineering implications of the proposed approach are illustrated through an application example wherein the proposed approach is utilized to generate a spatially varying undrained shear strength field for a two-dimensional plane strain slope, and the stability of the slope is analyzed by the finite element method with Monte-Carlo simulations. Since many material properties have a non-zero lower bound, the three-parameter gamma distribution is also discussed, and an asymptotically unbiased and consistent estimate of the lower bound is proposed.

1. Introduction

Spatial variability in material properties is typically simulated using a random field that is described by its marginal distribution and auto-correlation function (ACF). The marginal distribution of the random field is usually non-Gaussian, since most of the material properties have exclusively positive values. To achieve strictly positive values, various non-Gaussian distributions are commonly used, such as the lognormal distribution, the beta distribution and the gamma distribution (see [1–3]). Specifically, the lognormal and gamma distributions are often shifted to reflect a non-zero lower bound in material properties [4] that may be difficult to estimate. Furthermore, since the correlation of a material property at two separate points is expected to decrease as the distance of separation between the points increases, the ACF of a material property is likely to be a monotonically decreasing non-negative function of spatial distance, as seen in the triangular and exponential models.

The available approaches for generating these kinds of non-Gaussian random fields primarily utilize translation process theory, which applies a memoryless nonlinear transformation to an underlying Gaussian random field [5]. The so-called standard translation maps the Gaussian field $G(\mathbf{s};\omega)$ to the non-Gaussian field $X(\mathbf{s};\omega)$ through the marginal cumulative distribution function (CDF) as:

$$X(\mathbf{s};\,\omega) = F^{-1}\{\Phi[G(\mathbf{s};\,\omega)]\} \tag{1}$$

where Φ and F are, respectively, the marginal CDFs of a standard Gaussian variable and the target non-Gaussian variable; $\mathbf{s}$ is the coordinate vector; and ω is the random component. Note that the translation random field will be stationary if the underlying Gaussian field is stationary. An alternative class of translation methods expands the non-Gaussian field in terms of a set of orthogonal polynomials operating on an underlying Gaussian random field – referred to as polynomial chaos expansions (PCE). The standard translation technique in Eq. (1) is readily applicable in cases with any type of marginal distribution and ACF as long as they are compatible, whereas PCE approaches cannot match the marginal distribution exactly [6]. However, the ACF is often distorted by the standard translation, and various approaches have been developed to address the distortion in the ACF (e.g. [7,8]); most of these involve numerical and/or iterative methods. Common techniques for generating the underlying Gaussian

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field are the spectral representation method and the Karhunen–Loève expansion [9]. Both of these involve an infinite sum, which is unattainable in practice, so a truncated form is used as an approximation. As a result, the Gaussianity or stationarity may not be strictly guaranteed [10]; this may, in turn, affect the marginal distribution or stationarity of the translated non-Gaussian random field.

Various non-translation approaches have also been proposed using, for example, Markov theory, nonlinear filters and Ito calculus [11], spectral representation method with dependent variables [12], and recently higher-order expansions [13]. However, few of these approaches have been applied for simulating material properties in two- or three-dimensional models. In this regard, it is a difficult task to develop a general but easy to implement non-translation approach for non-Gaussian random fields. This study focuses on a specific type of random field for material properties; that is, the random field with a marginal gamma distribution. The gamma distribution has a very general form such that it includes many other distribution types. For example, the exponential distribution, chi-squared distribution, and Erlang distribution are all special cases of the gamma distribution. As a result, gamma random fields have the potential for relatively broad usage in practice. In their study of solving two-dimensional (2-D) elliptic stochastic PDEs, Wan and Karniadakis [14] showed that, without using the translation in Eq. (1), a homogeneous gamma random field can be obtained from several Gaussian random fields. They also derived an analytical relationship between the ACFs of the Gaussian and gamma random fields. Although Wan and Karniadakis' [14] approach is a translation-based approach (because it operates by summing the squares of several Gaussian random fields), it provides a hint of a conceptually simple non-translation approach, which will be elaborated in this study. The proposed approach takes advantage of the additive reproductive property of gamma random variables and has a simple geometric interpretation that makes it convenient for modelling material properties. The resulting random field possesses an arbitrary marginal gamma distribution and a decreasing ACF with a prescribed scale of fluctuation (SOF, denoted as θ) (see definition of SOF in [15]).

The paper is organized as follows. Section 2 gives an asymptotically unbiased and consistent estimate of the lower bound of a three-parameter gamma distribution; the estimation is shown to be physically reasonable. The algorithm of a 2-D gamma random field is proposed in Section 3. A generalized form for an m-dimension (m-D) gamma random field is described in Section 4. Section 5 briefly discusses extensions to the proposed methodology for anisotropic fields and to exercise greater control over the ACF. To illustrate its engineering applications, the proposed algorithm is used to simulate the spatial variability in the shear strength of soils in a plane strain slope in Section 6, where random field finite element analysis is used to evaluate slope stability.

2. Lower-bound estimation for the three-parameter gamma distribution

A random variable X follows the gamma distribution, if its probability density function (PDF), $f_X(x)$, takes the form:

$$f_X(x) = \begin{cases} \frac{(x-\gamma)^{\alpha-1}\cdot\exp[-(x-\gamma)/\beta]}{\beta^{\alpha}\cdot\Gamma(\alpha)}, & x > \gamma \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

in which γ is the lower bound of X; $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively; and $\Gamma(\cdot)$ is the gamma function. The gamma distribution is an attractive model for material properties because it possesses a clearly defined lower bound and can assume a variety of shapes from strongly skewed to asymptotically Gaussian when $\alpha\to\infty$. However, assigning an appropriate lower bound can be challenging. Prior to developing the new simulation method, we propose here a new strategy for estimating the lower bound.

2.1. Proposed method for lower bound estimation

The lower bound γ in Eq. (2) is commonly estimated using approaches such as maximum likelihood estimators, *method of moments*, and their extensions [16–18]. This section proposes an approach that is conceptually simple and rationally draws from the minima observed in the data. Its performance is then compared with the *method of moments* in the following subsection.

Let X_1, X_2, ..., X_n be a random sample of size n from a gamma distribution $F_X(x)$. For simplicity, this sequence is assumed to be ordered such that X_1 and X_n are the minimum and maximum of the sequence respectively, and the sample size n is reasonably large (e.g. larger than 30). It is assumed that the shape parameter α is known, or can be obtained by the method outlined in [19]. It can be demonstrated that the following estimate $\hat{\gamma}$ yields an asymptotically unbiased and consistent estimator of the lower bound γ:

$$\hat{\gamma} = X_1 - \alpha(X_2 - X_1) \tag{3}$$

The proof is similar to that for the lower bound of the beta distribution [2]. As the minimum of a sample, the CDF of X_1 is given by:

$$F_{X_1}(x) = [1 - F_X(x)]^n \tag{4}$$

For a reasonably large sample size n, Eq. (4) asymptotically leads to the Weibull distribution [20]:

$$F_{X_1}(x) = 1 - \exp\left\{-\left(\frac{\gamma - x}{\gamma - \kappa_1}\right)^k\right\} \tag{5}$$

in which κ_1 is the most probable minimum value, which can be taken as $F_X^{-1}\{1/(n+1)\}$ [17], and k is the shape parameter for the Weibull distribution that can be taken as α in Eq. (2) (see Appendix A). It should be noted that the random variable $(-X)$ also follows the gamma distribution with upper bound $-\gamma$. As such, for a reasonably large sample size n, the expectations of X_1 and X_2 may be estimated in a similar manner as those for X_n and X_{n-1} (see [20]):

$$E[X_1] = \gamma - \frac{\gamma - \kappa_1}{\alpha}\Gamma(\frac{1}{\alpha}) \tag{6}$$

$$E[X_2] = \gamma - (\gamma - \kappa_1)\frac{\Gamma(1/\alpha + 2)}{\Gamma(2)} \tag{7}$$

where $E[\cdot]$ is the expectation operator. Taking the expectation of both sides of Eq. (3) and utilizing the results in Eqs. (6) and (7), we observe that the estimator is unbiased:

$$E[\hat{\gamma}]=\gamma \tag{8}$$

Furthermore, as $n\to\infty$, both X_1 and X_2 converge in probability towards the lower bound γ, which implies that $\hat{\gamma}$ equals γ. Thus, $\hat{\gamma}$ is also a consistent estimator.

2.2. Comparison with the method of moments

In cases where α is known, the *method of moments* can be used to estimate the lower bound $\bar{\gamma}$ [16]:

$$\bar{\gamma} = \overline{X} - \sqrt{\alpha \times S} \tag{9}$$

where $\bar{\gamma}$ is the estimator of γ using the *method of moments* and $\overline{X}$ and S are respectively the sample mean and standard deviation. Taking the expectation of both sides of Eq. (9) shows that $\bar{\gamma}$ also yields an unbiased estimate for γ.

A comparison between the proposed method in Eq. (3) and the *method of moments* in Eq. (9) is conducted through Monte-Carlo simulations. First, a set of n gamma-distributed random numbers is generated. The parameters of the gamma distribution are listed as Case 1 in Table 1 and the PDF is shown in Fig. 1. The two methods are both

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