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Numerical solution of reliability models described by stochastic automata networks

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ABSTRACT

This paper presents the solution of Markov chain reliability models with a large state-space. To specify a system reliability model, we use our previously proposed methodology, which is based on the Stochastic Automata Networks formalism. We model parts of the system by arrowhead matrices with functional transition rates. As a result, the infinitesimal generator matrix of the reliability model has a distinctive structure. In this paper, we demonstrate that a block Gauss–Seidel method can be applied very efficiently to such a structure. The application of the proposed methodology is illustrated by an example of a standard 3/2 substation configuration. Even though its Markov chain reliability model has almost two million states, its steady-state probabilities can be estimated in just a few seconds of CPU time.

1. Introduction

Markov chains are one of the most often applied stochastic processes in system reliability models [1,2]. Application of Markov chains in reliability modelling include such diverse examples as analysis of software reliability [3], maintenance of road networks [4], and evaluation of reliability of small electronic components [5] or electrical power transmission systems [6]. As compared to the methods based on the independence of system components (e.g. fault tree analysis or reliability block diagrams), Markov models can be applied to systems with more complex behaviour, such as dependent failures or repairable items [7]. Moreover, unlike simulation-based techniques (e.g. a Monte Carlo method), Markov chain modelling is a stochastic analytic method which, in theory, can provide an exact solution. However, all the aforementioned techniques can be cumbersome and require many computational resources, especially if large systems and rare events are considered [8–11]. This is also true for Markov chains, which are rarely applied in reliability modelling of large systems due to an exponential growth of state-space. Therefore, most authors apply Markov chains for relatively small models, which can be tracked analytically [12–14]. In more complex cases, special software [15,16] is also applied for model creation and numerical solution [17,18]. However, for many real-life problems, the state-space of a Markov chain model can be estimated in millions. In such a case, the creation and storage of the transition matrix, as well as the calculation of steady-state probabilities require special attention.

Thus, efficient application of numerical methods is a crucial part of Markov chain modelling of large systems. However, it is still rarely addressed in reliability modelling studies, and only in a few exceptions can one find examples of steady-state solutions of reliability models whose state-space exceeds a million [19]. Thus, it seems that well-established methods of numerical solution of Markov chain models [20] have not yet been extensively applied in reliability modelling studies. So far, the most often used approaches in dealing with large Markov models are various state-space reduction strategies [21]. Among them, we can mention an exclusion of redundant system states and truncation of states with low probabilities [22,23], state aggregation and lumping methods [24,25] or some mixture of both these techniques [26]. Another similar approach is to decompose a large system into smaller, nearly independent components, and to solve the smaller models separately [27,28]. However, these techniques are by definition approximations and exhibit inherent solution errors. Only the solution of the complete Markov model can provide an exact (as far as computational precision allows) solution, thus evaluating all possible scenarios, including all rare events.

In this study, we address a numerical solution of complete Markov chain models, described by the Stochastic Automata Networks (SANs) formalism [29]. The SAN formalism applies Kronecker algebra operations to store a transition matrix of the Markov chain model in a compact form, thus mitigating the problem of dimensionality. However, a steady-state solution remains a serious problem because Kronecker's algebra approach requires a more sophisticated application of standard numerical methods [30].

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So far, most examples of SANs applications considered queuing system models [31–33]. The use of SANs in system reliability modelling is still relatively rare [34,35] and, to our knowledge, the published studies did not consider a steady-state probability calculation. In our previous paper [36], we presented a methodology for power system reliability specification using the SAN formalism. We demonstrated that various reliability scenarios could be described using arrowhead matrices and functional transition rates. As a result, the transition rate matrices of the created Markov models have a distinctive structure and we proposed that it could be advantageous for an efficient estimation of steady-state probabilities. However, we did not address this problem in the previous study because for the relatively small models presented in [36] it was possible to generate the full transition matrix from the SAN descriptor and to estimate steady-state probabilities using standard direct algorithms, such as Gaussian elimination. For larger models, such an approach would not be feasible due to operating memory constraints. In that case, one must apply various iterative techniques for calculation of steady-state probabilities, and we address this problem in this study.

Our research showed that the distinctive structure of the created reliability models allows us to apply a block Gauss–Seidel method very efficiently. That is, due to the high fidelity of system components, steady-state solution of power systems described by arrowhead matrices converges very rapidly when using the Gauss–Seidel algorithm. In addition, the specific block structure arising from the SAN formalism is very suitable for block-iterative methods. Moreover, the analytical research showed that inner iteration of the block Gauss–Seidel method can be solved very efficiently. Numerical experiments supported the analytical results and the block Gauss–Seidel method outperformed other standard methods in solving steady-state probabilities of a representative Markov model of a 3/2 substation configuration with nearly two million states.

The structure of this paper is as follows. An introduction to the steady-state solution of Markov chain models is presented in Section 2. In Section 3, we explain the choice of the block Gauss–Seidel method for the steady-state solution, based on the structural properties of the model transition matrix. In Section 4, we present the efficient implementation of the block Gauss–Seidel method specially modified for models described by the SAN formalism and arrowhead matrices. In Section 5, we present the SAN specification of the 3/2 substation configuration using the methodology proposed in [36]. This model is later used in Section 6 as a case study to illustrate the efficiency of the proposed implementation of the block Gauss–Seidel method.

2. Steady-state solution of Markov chain models

In this section, we briefly introduce the theoretical background and numerical solution methods for steady-state calculations of Markov chain models. For more information, we refer the reader to [20,37].

2.1. Estimation of steady-state probabilities

Most examples in Markov chain reliability modelling consider ergodic continuous time Markov chains with a finite number of system states. Basically, such a system can be defined by its transition rate matrix (i.e. an infinitesimal generator) $\mathbf{Q}$. In reliability models, transition rates would mainly describe failure and repair rates of various system components. A steady-state transition (row) vector $\boldsymbol{\pi}$ can be estimated from the following equation

$$\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0}, \tag{1}$$

where $\mathbf{0}$ denotes a row vector consisting of zeros.

An ergodic Markov chain $\mathbf{Q}$ is a singular matrix, i.e. its rank is equal to $(n-1)$ if n is the size of the state-space. Thus, an additional condition is used

$$\boldsymbol{\pi} \cdot \mathbf{e} = 1, \tag{2}$$

where $\mathbf{e}$ denotes a column vector consisting of ones. Basically, it means that the sum of all steady-state probabilities is equal to 1.

It can be seen from (1) and (2) that computation of steady-state probabilities is basically a solution of a linear system of equations, or, equivalently, the computation of a left-side eigenvector. While it is not a particularly difficult task for a small infinitesimal generator $\mathbf{Q}$, problems arise when the size of the matrix $\mathbf{Q}$ is counted in thousands or even millions.

There are three big classes of algorithms for the calculation of steady-state probabilities: direct methods, iteration methods and projection methods.

1) **Direct methods** compute a solution of a system of linear equations in a fixed number of operations. In theory, they will compute an exact solution, if there are no round-off errors. The best-known direct method is the classical Gaussian elimination algorithm and its variations, such as LU decomposition. The main drawback of the direct methods is a fill-in of zero elements during the reduction phase, which makes them incompatible with sparse and compact storage schemes. Such an alteration of the infinitesimal generator matrix makes direct methods inapplicable for large problems because of computer memory constraints and a build-up of rounding errors. However, direct methods can be successfully applied for relatively small problems (less than 10,000 states) or some specific problems, for example, when banded storage schemes can be implemented.
2) **Iterative methods** are based on the property of successive convergence to the desired solution from an initial iteration vector. The most popular iterative algorithms are the methods of Jacobi, Gauss–Seidel, successive over-relaxation (SOR) and the power method.
   The main advantage of iterative algorithms, as compared with direct methods, is that an infinitesimal generator matrix is not transformed during the solution. This means that sparse storage schemes can be successfully implemented, which extends the number of states to at least hundreds of thousands. In addition, it prevents the build-up of rounding errors, which increases the stability of the algorithms. It is also worth mentioning that the implementation of standard iterative methods is relatively easy (especially compared with more advanced projective methods) and it does not require a deep knowledge of linear algebra.
   The main disadvantage of iterative methods is that convergence to the solution is not always guaranteed or it can be very slow, especially for practical engineering and industrial problems.
3) **Projection methods** are advanced iterative techniques, based on the approximation of an exact solution by a relatively small substate of vectors. Some methods theoretically will compute an exact solution in n steps, but practically a desirable solution can be found in a much smaller number of steps. The projection methods include Krylov subspace methods, such as the Arnoldi algorithm or generalised minimal residuals (GMRES) method for eigenvalues. Conjugate gradient squared (CGS), biconjugate graduate stabilized (BiCGStab) and transpose-free quasi-minimal residuals (TFQMR) are also among suitable methods to calculate steady-state probabilities.

Projection methods can be successfully applied for solving large systems of linear equations, but some problems exist. For example, the use of projection methods does not guarantee regular convergence and the methods can sometimes break down. Moreover, as for iterative methods, the convergence rate can be very slow for realistic engineering and industrial applications.

Thus, a variety of different techniques can be applied for the calculation of steady-state probabilities, but there are no general rules on how to find the most suitable method for a given problem. However, if an algorithm somehow ‘considers’ the special properties of a Markov chain model structure, it can significantly outperform other numerical methods [38]. Therefore, an experimental and analytical investigation is needed for each distinct type of problem.