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Analysis of rectangular cracks in elastic bodies

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ABSTRACT

The solution of Volterra dislocation is derived in an infinite elastic body. Stress components are Cauchy singular at dislocation location. The stress field is utilized to construct integral equations for rectangular cracks with arbitrary arrangement in the body. The solution to integral equations is used to determine stress intensity factors on the crack edges. Numerical results are presented for a rectangular crack under various loads. Furthermore, interaction between two cracks, having a line of symmetry, is studied.

1. Introduction

The two-dimensional analysis of rectangular cracks with aspect ratio around unity leads to erroneous results. In three-dimensional (3-D) analysis of cracks, all three displacement components are coupled in the Navier's equations; thus, in general, mixed mode fracture occurs. This is in contrast with the two-dimensional (2-D) situation, wherein the governing equations of anti-plane and in-plane deformations decouple. Weaver [1], adjusted the basic equations of the boundary integral equation method to derive integral expressions for the stress distribution on the surface of a 3-D crack. The integral equations were solved numerically for rectangular cracks and stress intensity factors (SIFs) were obtained for opening mode and also under uniform in-plane shear traction. Kassir determined stress intensity factors for a narrow rectangular crack under mode I [2] and uniform in-plane shear [3]. Dynamic SIFs were obtained for three coplanar rectangular cracks subjected to time-harmonic uniform normal traction by Itou [4]. Based on the body force method Wang et al. derived hyper-singular integral equations for the opening mode of a rectangular crack [5]. The solution of integral equations is utilized to determine SIF along a length of cracks with different aspect ratios. Itou analyzed two coplanar rectangular cracks situated in the thickness direction of an elastic infinite plate under normal impact load [6]. Liu and Zhou, investigated the opening mode of a rectangular crack in an orthotropic body [7]. Three pairs of dual integral equations were derived and solved by the Schmidt method. The 3-D finite element formulation was utilized by Jin and Wang for the analysis of elastic single edge cracked sheets [8]. The SIFs, in-plane, and out of plane T-stresses in front of the crack were determined for a wide range of the geometries. A 3-D displacement discontinuity method was used by Wu and Olson to formulate the problem of multiple interacting cracks [9]. The numerical solution to these equations led to displacement discontinuity on a crack surface.

In this study, closed form solutions are derived for stress fields in an infinite elastic body containing Volterra dislocation. Stress components exhibit the familiar Cauchy type singularity at dislocation location. The solution is used to construct integral equations for the density of dislocations on the surface of a rectangular crack. The stress intensity factors (SIFs), on the edges of a rectangular crack, are derived in terms of the density of dislocations. The integral equations are solved numerically, thereby obtaining SIFs of a crack. Formulation is generalized for the analysis of multiple interacting rectangular cracks. The geometry of cracks and loading are such that crack closing does not occur. In two examples, interaction between two cracks is investigated.

2. Dislocation formulation

The constitutive equations in linear elasticity i.e., Hooke's law, for an isotropic material with Poisson's ratio ν and shear modulus of elasticity μ, are

$$\boldsymbol{\sigma} = 2\mu\left[\frac{\nu}{1-2\nu}\mathrm{tr}(\varepsilon)\mathbf{I} + \boldsymbol{\varepsilon}\right] \tag{1}$$

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where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, respectively, and $\boldsymbol{I}$ is the unit matrix. Ignoring body forces we substitute the above equations into equations of equilibrium, $\operatorname{div}\boldsymbol{\sigma} = 0$, to arrive at the Navier's equations

$$\frac{1}{1-2\nu}\,\operatorname{grad}\operatorname{div}\boldsymbol{u} + \Delta\boldsymbol{u} = 0 \tag{2}$$

where $\boldsymbol{u} = (u_x, u_y, u_z)$ is the displacement vector and Δ is the Laplace's operator. We consider a Volterra dislocation located at the origin of coordinates wherein the dislocation cut is the first-quadrant of the xy-plane. The Burgers vector of dislocation is identified by its components, B_x, B_y, and B_z. The Volterra dislocation may be represented as

$$\begin{aligned} &u_i(x,y,0^+) - u_i(x,y,0^-) = B_i H(x)H(y) \\ &\sigma_{iz}(x,y,0^+) = \sigma_{iz}(x,y,0^-), \quad i \in \{x,y,z\} \end{aligned} \tag{3}$$

where $H(\cdot)$ is the Heaviside step-function. Application of the complex Fourier transforms with respect to variables x and y to Eq. (2), results in

$$\begin{aligned} &U_{,zz} - i\frac{\alpha}{1-2\nu}W_{,z} - \left[\frac{2(1-\nu)}{1-2\nu}\alpha^2 + \beta^2\right]U - \frac{\alpha\beta}{1-2\nu}V = 0 \\ &V_{,zz} - i\frac{\beta}{1-2\nu}W_{,z} - \frac{\alpha\beta}{1-2\nu}U - \left[\alpha^2 + \frac{2(1-\nu)}{1-2\nu}\beta^2\right]V = 0 \\ &W_{,zz} - i\frac{\alpha}{2(1-\nu)}U_{,z} - i\frac{\beta}{2(1-\nu)}V_{,z} - \frac{1-2\nu}{2(1-\nu)}(\alpha^2+\beta^2)W = 0 \end{aligned} \tag{4}$$

In Eqs. (4), $i = \sqrt{-1}$, subscript z following a comma designates derivative with respect to variable z, and the Fourier transformations of displacement components are

$$\{U,V,W\}(\alpha,\beta,z) = \mathcal{F}^2\left[u_x(x,y,z), u_y(x,y,z), u_z(x,y,z); \alpha, \beta\right] \tag{5}$$

Furthermore, the Fourier transforms of conditions (3) become

$$\begin{aligned} &U(\alpha,\beta,0^+) - U(\alpha,\beta,0^-) = B_x\Lambda \\ &V(\alpha,\beta,0^+) - V(\alpha,\beta,0^-) = B_y\Lambda \\ &W(\alpha,\beta,0^+) - W(\alpha,\beta,0^-) = B_z\Lambda \\ &U_{,z}(\alpha,\beta,0^+) - U_{,z}(\alpha,\beta,0^-) = i\alpha B_z\Lambda \\ &V_{,z}(\alpha,\beta,0^+) - V_{,z}(\alpha,\beta,0^-) = i\beta B_z\Lambda \\ &W_{,z}(\alpha,\beta,0^+) - W_{,z}(\alpha,\beta,0^-) = \frac{i\nu}{1-\nu}\{\alpha B_x + \beta B_y\}\Lambda \end{aligned} \tag{6}$$

where

$$\Lambda = \mathcal{F}^2[H(x)H(y); \alpha; \beta] = \pi^2\delta(\alpha)\delta(\beta) - \frac{1}{\alpha\beta} + i\pi\left[\frac{\delta(\alpha)}{\beta} + \frac{\delta(\beta)}{\alpha}\right] \tag{7}$$

In Eq. (7), $\delta(\cdot)$ is the Dirac delta function. The solution to Eqs. (4) satisfying (6) are

$$\begin{aligned} &U(\alpha,\beta,z) = \frac{-e^{\lambda z}}{4\lambda(1-\nu)}\{B_x[2(1-\nu)\lambda + \alpha^2 z] + B_y\alpha\beta z + iB_z\alpha(1-2\nu+\lambda z)\}\Lambda \\ &V(\alpha,\beta,z) = \frac{-e^{\lambda z}}{4\lambda(1-\nu)}\{B_x\alpha\beta z + B_y[2(1-\nu)\lambda + \beta^2 z] + iB_z\beta(1-2\nu+\lambda z)\}\Lambda \\ &W(\alpha,\beta,z) = \frac{e^{\lambda z}}{4\lambda(1-\nu)}\{i(B_x\alpha + B_y\beta)(1-2\nu-\lambda z) - B_z\lambda[2(1-\nu)-\lambda z]\}\Lambda, \quad z<0 \\ &U(\alpha,\beta,z) = \frac{e^{-\lambda z}}{4\lambda(1-\nu)}\{B_x[2(1-\nu)\lambda - \alpha^2 z] - B_y\alpha\beta z - iB_z\alpha(1-2\nu-\lambda z)\}\Lambda \\ &V(\alpha,\beta,z) = \frac{e^{-\lambda z}}{4\lambda(1-\nu)}\{-B_x\alpha\beta z + B_y[2(1-\nu)\lambda - \beta^2 z] - iB_z\beta(1-2\nu-\lambda z)\}\Lambda \\ &W(\alpha,\beta,z) = \frac{e^{-\lambda z}}{4\lambda(1-\nu)}\{i(B_x\alpha + B_y\beta)(1-2\nu+\lambda z) + B_z\lambda[2(1-\nu)+\lambda z]\}\Lambda, \quad z>0 \end{aligned} \tag{8}$$

where $\lambda = \sqrt{\alpha^2+\beta^2}$. In view of Eqs. (1), (8) and with the aid of integral formulas given in Appendix (A.1), stress components caused by the Volterra dislocation, in the infinite-space, yield

$$\begin{aligned}
\sigma_{xx}(x,y,z) = {}& \frac{\mu}{4\pi(1-\nu)}\Bigg(B_x\frac{z}{x^2+z^2}\left[\frac{3x^2+z^2}{x^2+z^2} + \frac{2y}{r} + \frac{y[x^2(r^2+x^2) - z^2(r^2-x^2)]}{r^3(x^2+z^2)}\right] + B_y\frac{z}{r}\left[2\nu\frac{r+x}{y^2+z^2} - \frac{x}{r^2}\right] \\
&- B_z\left\{\frac{x}{x^2+z^2}\left[\frac{x^2-z^2}{x^2+z^2} + \frac{y}{r} - \frac{yz^2(3r^2-y^2)}{r^3(x^2+z^2)}\right] + \frac{2\nu y(r+x)}{r(y^2+z^2)}\right\}\Bigg) \\
\sigma_{yy}(x,y,z) = {}& \frac{\mu}{4\pi(1-\nu)}\Bigg(B_y\frac{z}{y^2+z^2}\left[\frac{3y^2+z^2}{y^2+z^2} + \frac{2x}{r} + \frac{x[y^2(r^2+y^2) - z^2(r^2-y^2)]}{r^3(y^2+z^2)}\right] + B_x\frac{z}{r}\left[2\nu\frac{r+y}{x^2+z^2} - \frac{y}{r^2}\right] \\
&- B_z\left\{\frac{y}{y^2+z^2}\left[\frac{y^2-z^2}{y^2+z^2} + \frac{x}{r} - \frac{xz^2(3r^2-x^2)}{r^3(y^2+z^2)}\right] + \frac{2\nu x(r+y)}{r(x^2+z^2)}\right\}\Bigg) \\
\sigma_{zz}(x,y,z) = {}& -\frac{\mu}{4\pi(1-\nu)}\Bigg\{B_x\frac{z}{x^2+z^2}\left[\frac{x^2-z^2}{x^2+z^2} - \frac{y[z^2(r^2+z^2) - x^2(r^2-z^2)]}{r^3(x^2+z^2)}\right] \\
&+ B_y\frac{z}{y^2+z^2}\left[\frac{y^2-z^2}{y^2+z^2} - \frac{x[z^2(r^2+z^2) - y^2(r^2-z^2)]}{r^3(y^2+z^2)}\right] \\
&+ B_z\Bigg[\frac{x(x^2+3z^2)}{(x^2+z^2)^2} + \frac{y(y^2+3z^2)}{(y^2+z^2)^2} + \frac{x^3y^3(x^2+y^2)^2}{r^3(x^2+z^2)^2(y^2+z^2)^2} \\
&+ \frac{xyz^2\left[3(r^2-z^2)^3 + 11z^2(r^2-z^2)^2 + 8z^4(2r^2-z^2)\right]}{r^3(x^2+z^2)^2(y^2+z^2)^2}\Bigg]\Bigg\} \\
\sigma_{yz}(x,y,z) = {}& -\frac{\mu}{4\pi(1-\nu)}\Bigg(B_y\left\{\frac{y}{y^2+z^2}\left[\frac{y^2-z^2}{y^2+z^2} + \frac{x}{r} - \frac{xz^2(3r^2-x^2)}{r^3(y^2+z^2)}\right] + (1-\nu)\frac{x(r+y)}{r(x^2+z^2)}\right\} - B_x\frac{1}{r}\left[\nu - \frac{z^2}{r^2}\right] \\
&+ B_z\frac{z}{y^2+z^2}\left[\frac{y^2-z^2}{y^2+z^2} - \frac{x[z^2(r^2+z^2) - y^2(r^2-z^2)]}{r^3(y^2+z^2)}\right]\Bigg) \\
\sigma_{xz}(x,y,z) = {}& -\frac{\mu}{4\pi(1-\nu)}\Bigg(B_x\left\{\frac{x}{x^2+z^2}\left[\frac{x^2-z^2}{x^2+z^2} + \frac{y}{r} - \frac{yz^2(3r^2-y^2)}{r^3(x^2+z^2)}\right] + (1-\nu)\frac{y(r+x)}{r(y^2+z^2)}\right\} - B_y\frac{1}{r}\left[\nu - \frac{z^2}{r^2}\right] \\
&+ B_z\frac{z}{x^2+z^2}\left[\frac{x^2-z^2}{x^2+z^2} - \frac{y[z^2(r^2+z^2) - x^2(r^2-z^2)]}{r^3(x^2+z^2)}\right]\Bigg) \\
\sigma_{xy}(x,y,z) = {}& \frac{\mu}{4\pi(1-\nu)}\left\{B_x\frac{z}{r}\left[(1-\nu)\frac{r+x}{y^2+z^2} - \frac{x}{r^2}\right] + B_y\frac{z}{r}\left[(1-\nu)\frac{r+y}{x^2+z^2} - \frac{y}{r^2}\right] + B_z\frac{1}{r}\left[(1-2\nu) - \frac{z^2}{r^2}\right]\right\}
\end{aligned} \tag{9}$$

where $r = \sqrt{x^2+y^2+z^2}$. From Eqs. (9), one may observe that stress components are Cauchy singular at the dislocation location. Moreover, on the plane, $z = 0$ only the climb component of Burgers vector B_z takes part in the expressions for stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ and σ_{xy}, whereas the glide components B_x and B_y induce sheer stresses σ_{xz} and σ_{yz}.

3. Integral equations for cracks

The dislocation solution (9) may be utilized for the solution of crack problems. We consider a body weakened by a rectangular crack having dimensions, $2a \times 2b$. The origin of Cartesian coordinates system xyz is located at the center of the crack; thus $-a < x < a$ and $-b < y < b$. Let Volterra dislocations with density functions b_x, b_y, and b_z be distributed on the crack surface. Dislocation densities are defined as

$$b_x = \frac{\partial^2}{\partial x\partial y}(u^+ - u^-), \quad b_y = \frac{\partial^2}{\partial x\partial y}(v^+ - v^-), \quad b_z = \frac{\partial^2}{\partial x\partial y}(w^+ - w^-) \tag{10}$$

where superscripts $(+)$ and $(-)$ designate, respectively, the upper and lower surfaces of a crack. By virtue of Eqs. (9), stress components in the body due to distribution of dislocations are