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Generation of random stress tensors

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ABSTRACT

To correctly incorporate stress variability in the increasingly widespread application of probabilistic-related rock mechanics analyses, a robust approach for random stress tensor generation is essential. However, currently, the customary scalar/vector approaches to the generation of random stress tensors, which violate the tensorial nature of stress, together with other existing quasi-tensorial applications that consider the tensor components as statistically independent variables, may yield biased results. Here, we propose a multivariate random vector generation approach for generating random stress tensor components that is based on tensorial techniques and which incorporates inter-component correlation. Differences between the proposed fully tensorial and existing quasi-tensorial approaches are demonstrated by examining the distributions of the tensors generated using both approaches, and the efficacy and transformational consistency of the proposed fully tensorial approach are investigated by generating random tensors in different coordinate systems. Our results suggest application of the existing quasi-tensorial approach (which ignores covariance) leads to greater scatter in generated tensors than does application of the proposed fully tensorial approach (which includes covariance). Additionally, the transformational consistency of the proposed fully tensorial approach allows generation of random tensors in any convenient coordinate system, while the existing quasi-tensorial approach only permits generation of random tensors in a particular coordinate system. The proposed fully tensorial approach provides a method that will assist with probabilistic-related analyses of rock engineering structures.

1. Introduction

In situ stress is an important parameter for a wide range of endeavours in rock mechanics, including rock engineering design, hydraulic fracturing analysis, rock mass permeability and evaluation of earthquake potential.[1–5] Because of the inherent complexity of fractured rock masses in terms of varying rock properties, the presence of discontinuities and unclear boundary conditions,[4] stress in rock often displays significant variability.[6] With the increasingly widespread application of probabilistic or reliability-based analyses in rock mechanics, incorporating stress variability in these analyses is becoming a necessity.[7–12] A robust approach for random stress tensor generation – i.e. one that is faithful to the tensorial nature of stress – is essential for such work. Here, and particularly to assist probabilistic-related analyses in rock mechanics that need to consider the inherent variability of *in situ* stress, we present a fully tensorial technique for generating random stress tensors.

Currently in rock mechanics, stress magnitude and orientation are customarily processed separately (e.g. Fig. 1). This processing effectively decomposes the second order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, to which classical statistics[13] and directional statistics[14], respectively, are applied.[6,7,15–26] Following this, probabilistic analyses are generally implemented by drawing random variates separately from the statistical distributions of both principal stress magnitude and orientation.[7] These customary scalar/vector approaches violate the tensorial nature of stress and may yield biased results.[27–30] In particular, orthogonality of the randomly generated principal stresses is not guaranteed.

Rather than analysing principal stress magnitude and orientation separately, and in order to remain faithful to the tensorial nature of stress, stress analysis should be conducted on the basis of tensor components obtained in a common Cartesian coordinate system. Several researchers have followed this technique in random stress tensor generation,[31–33] with the random tensors being based on the mean and variance of each tensor component relative to a common coordinate system. However, this existing quasi-tensorial approach considers the tensor components as statistically independent variables, and ignores any correlation between them. The result is, to date there seems to have been no mathematically rigorous proposal from the rock mechanics community for random stress tensor generation.

Stress tensors, which are 2×2 or 3×3 symmetric matrices, together

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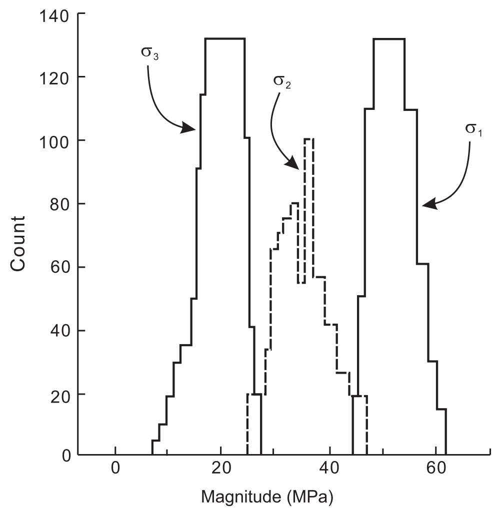
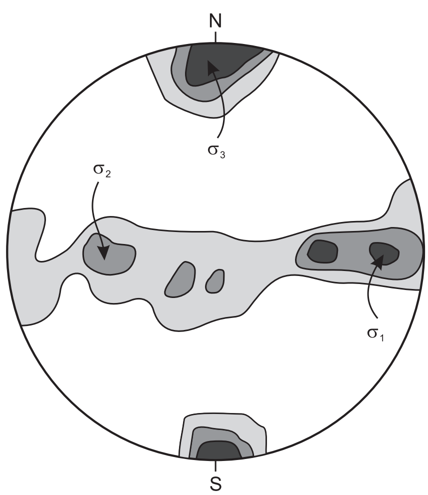


(a) Distribution of principal stress magnitudes



(b) Contouring of principal stress orientations

Fig. 1. Customary analyses of stress examine principal stress magnitude and orientation separately using classical statistics and directional statistics, respectively (after Brady & Brown[26]).

with other matrix-valued quantities, play a pivotal role in many subjects such as solid mechanics, physics, earth science, medical imaging and economics.[34] To explicitly account for the inherent variability of such matrix-valued quantities, matrix variate statistics – as a generalisation of multivariate statistics – has been developed,[34] and we have previously demonstrated that this statistics is appropriate for stress variability analysis.[35] Matrix variate statistics and multivariate statistics are often used interchangeably by statisticians,[36–38] and it has been demonstrated that matrix variate analysis of stress tensors and multivariate analysis of their distinct components are statistically equivalent.[34,35,39] Thus, from the viewpoint of random tensor generation, instead of generating the whole stress tensor, it is correct and more convenient to form a stress tensor by generating the distinct tensor components in a multivariate manner. Here, we use "distinct tensor components", rather than the customary "independent tensor components", and the reason for this is discussed later.

In the present paper, in order to propose a robust method for generating random stress tensors, we first examine related work in rock mechanics. We discuss the deficiency of processing principal stress magnitude and orientation separately, and the inappropriateness of the customary scalar/vector random stress generation approach, and examine the applicability of the existing quasi-tensorial applications found in the literature. Then, using a multivariate normal distribution model of the distinct tensor components as an example, we present a multivariate random vector generation approach for generating random stress tensor components that incorporates inter-component correlation. We illustrate the difference between the existing quasi-tensorial and new approaches, and by analysing actual stress data we demonstrate the efficacy of the proposed fully tensorial approach by examining the distributions of the tensors generated using both approaches in terms of tensor components and principal stresses. Finally, the transformational consistency of the proposed fully tensorial approach is illustrated by generating random tensors in different coordinate systems.

2. Related work

2.1. Deficiency of the customary scalar/vector approach

As noted above, the customary scalar/vector approach employed in rock mechanics of processing principal stress magnitude and orientation separately may yield unreasonable results. Here, we re-present the succinct and clear example presented in Dyke et al.[29] to emphasise this.

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the two stress states

$$\mathbf{S}_1 = \begin{bmatrix} 18 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } \mathbf{S}_2 = \begin{bmatrix} 10 & 0 \\ 0 & 18 \end{bmatrix}, \quad (1)$$

referred to a common Cartesian coordinate system. These stress states are also represented in Fig. 2a by ellipses whose semi-axes denote the magnitude and orientation of their principal values. $\mathbf{S}_1$ and $\mathbf{S}_2$ clearly possess identical principal stress magnitudes, but different principal stress orientations. If we separately determine the principal stress mean magnitudes and mean orientations, the result is a mean represented by ellipse A shown in Fig. 2b. However, when the tensorial approach that averages the corresponding tensor components[40] is applied, the mean symbolised by ellipse B results (Fig. 2b). If we apply the principles of solid mechanics and consider $\mathbf{S}_1$ and $\mathbf{S}_2$ as perturbations from some mean state, then the tensorial mean of

$$\bar{\mathbf{S}} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} \quad (2)$$

is clearly correct. In essence, the customary scalar/vector approach is deficient in that it averages stress states (i.e. principal stresses) that are referred to their own, potentially unique, coordinate systems. This both violates the tensorial nature of stress and erroneously applies statistical tools to process data that are referred to different geometrical bases.

Although this example concerns only the case of calculating the mean of two stresses, this fundamental reasoning applies also to the case of additional stress tensors, as well as to statistics such as dispersion calculation, distribution characterisation and generation of random stress tensors. Thus, the conclusion to be drawn is that statistical and probabilistic applications based on separate processing of principal stress magnitude and orientation will be incorrect and may yield unreasonable results. Since the generation of random stress tensors depends on the underlying statistical model, randomly generating stress magnitude and orientation separately will be inappropriate. Instead, random stress tensors should be generated using tensorial approaches that generate random tensor components referred to a common Cartesian coordinate system, and, as shown below, some reports of this exist in the literature.

2.2. Existing quasi-tensorial random tensor generation approaches

A survey of the rock mechanics literature reveals the existence of a tensorial approach to random tensor generation that is based on the means and variances of the distinct tensor components of measured *in situ* stress data, and which generates random tensors in the coordinate system that aligns with the direction of the principal components of the