International Journal of Rock Mechanics & Mining Sciences 96 (2017) 23–33

Contents lists available at ScienceDirect

International Journal of Rock Mechanics & Mining Sciences

journal homepage: www.elsevier.com/locate/ijrmms

Analysis of wave velocity anisotropy of rocks using ellipse fitting

Andrea Ferreira Borges*

Department of Geology, Federal University of Rio de Janeiro, Brazil

ARTICLE INFO

Keywords:
Ellipse fitting
Anisotropic rock
Wave propagation velocity

1. Introduction

Most rocks show some degree of anisotropy in their properties such as strength, elastic constants, permeability and propagation velocities of volumetric (compressional and shear) waves. Anisotropic rock properties are influenced by foliation, schistosity, layering, bedding and cleavage, as anisotropic behavior is related to the rock fabric, but it can also be due to the presence of microcracks, joints and faults or induced by stress field and stress history.[1,2] Investigation on anisotropic rocks has many applications including foundation engineering, tunnelling, mining, petroleum engineering, and hydrogeology.[2]

Some directional properties such as permeability and wave velocities are usually described as symmetric second rank tensors, and can be visualized as ellipsoids (or ellipses, in two dimensions). Semi-axes are equal to, or a function of, the principal values of the property being analysed. The aim of this study is to present a method for least-squares based geometric fitting of ellipses in polar coordinates, which is intended to be applied to two-dimensional analysis of anisotropic properties of rocks and other materials which are measured as azimuthal data. Since these are usually noisy data, ellipse fitting is a useful tool for estimating principal values (maximum and minimum) and their directions in two dimensional studies.

Measuring wave velocities provides an indirect investigation of the material, revealing information about rock structure not accessible by direct observation. For instance, in jointed or interbedded rocks, maximum compressional wave velocities are observed parallel to joint orientation or bedding planes.[2–4] Furthermore, wave velocities can be statistically correlated to other rock properties such as strength and permeability, and can also be used for rock mass classification purposes.[2,5] Numerous experimental studies are available relating wave velocities to rock fabric, porosity, microcracks and joints,[6–11] stress state,[6,9,12–14] permeability,[4,15–17] and fluid saturation.[18,19]

Additionally, compressional and shear wave velocity tensors are related to elastic constants of anisotropic rocks. Transverse isotropy model is typically used to describe bedded sedimentary rocks, rocks with aligned microcracks, or rock masses with one set of fractures. This model is characterized by five independent elastic constants, which can be calculated from rock bulk density and compressional (or primary) and shear (or secondary) velocities, denoted by V_p and V_s respectively, measured in specific directions. Two-dimensional analysis is suitable for transversely isotropic rocks, where directions normal and parallel to bedding planes are assumed to be the principal directions, then V_p and V_s measured in these two directions allow calculation of four elastic constants, and at least one measurement at an oblique angle is required to calculate the fifth elastic constant, for which compressional velocity at 45° ($V_{p(45o)}$) is usually chosen.[9,20–23] In many studies of transversely isotropic rocks,[6,7,14,23,24] V_p and V_s are measured in two directions, normal and parallel to the bedding planes, assumed to be the principal velocities. In these cases, V_p at an oblique angle may be calculated from the ellipse equation.

Sedimentary rocks may show relatively high degree of velocity anisotropy, as shown by Vernik and Liu [21] from measurements in samples of kerogen-rich shales. The authors observed that these rocks have strong anisotropy, as both compressional and shear velocities parallel to bedding planes are very high compared to the velocities normal to bedding, reaching a ratio of 1.47 between maximum and minimum values of both V_p and V_s.

2. Wave velocity anisotropy

In anisotropic media, not only compressional and shear wave velocities depend on direction of propagation, but polarization of the shear wave occurs, so three wave modes can propagate, one compres-

* Corresponding address: Universidade Federal do Rio de Janeiro IGEO/Departamento de Geologia, Av. Athos da Silveira Ramos, 274, sl. J0-05A, CCMN, Cidade Universitária, Ilha do Fundão, Rio de Janeiro, RJ 21941-972, Brazil.

E-mail address: aborges@geologia.ufrj.br.

http://dx.doi.org/10.1016/j.ijrmms.2017.04.013
Received 4 August 2016; Received in revised form 4 April 2017; Accepted 23 April 2017
1365-1609/

sional and two shear waves.[1] A detailed discussion of wave propagation in anisotropic media is presented, for instance, in 1,25,26. This study concerns only compressional velocity analyses.

Anisotropic wave velocities are usually represented in polar graphs, which are a convenient way of observing principal directions and anisotropy ratios. Some authors choose to plot velocity in the radial axis,[10,18,27,28] others prefer to plot squared velocity [3] or even slowness, which is the inverse of velocity.[11,29]

Budavari [27] presented a mathematical method for determining V_p based on the characterization of the rock mass anisotropy by an ellipsoidal type velocity law. He described V_p tensor as an ellipsoid with semi-axes equal to the principal values of V_p; the same approach was also used in 28:

$$\frac{X^2}{V_{pX}^2}+\frac{Y^2}{V_{pY}^2}+\frac{Z^2}{V_{pZ}^2} = 1 \tag{1}$$

where X, Y, and Z are the principal directions and V_{pX}, V_{pY}, and V_{pZ} are the principal compressional wave velocities.

In two-dimensional analysis of wave propagation (e.g. in the XY plane), anisotropic V_p is represented by an ellipse with semi-axes V_{pX} and V_{pY}:

$$\frac{X^2}{V_{pX}^2}+\frac{Y^2}{V_{pY}^2} = 1 \tag{2}$$

Oda et al. [3] chose to represent velocity anisotropy as a function of the squared velocity rather than the velocity, because $(V_p)^2$ is directly related to the elastic constants and because the ratio between the squared measured velocity and the squared reference velocity in non-cracked rock $(V_p/V_{p0})^2$ is related to the Rock Quality Designation defined in 30. Supposing $(V_p)^2$ can be approximated by an ellipse:

$$\frac{X^2}{V_{pX}^4}+\frac{Y^2}{V_{pY}^4} = 1 \tag{3}$$

Bloch et al. [29] suggested representing the slowness $(1/V_p)$ in radial plots for estimating in situ stress orientation, because the direction of maximum slowness indicates maximum horizontal in situ stress based on the orientation of microcracks resulting from stress relief. Benson et al. [11] applied the general equation for an ellipsoid to the slownesses to analyse experimental data. This interpretation indicates that the slowness may be approximated by an elliptic function:

$$V_{pX}^2X^2+V_{pY}^2Y^2 = 1 \tag{4}$$

In the present study, four hypoteses concerning the variable used to define the ellipse equation were considered: (1) velocity (V_p); 2) squared velocity $(V_p)^2$; (3) slowness $(1/V_p)$, and (4) squared slowness $(1/V_p)^2$. The method of ellipse fitting described in Section 3 was used to fit these four variables and to statistically evaluate and compare these hypoteses.

3. Proposed method for ellipse fitting

Several authors describe ellipse-fitting methods for various applications, including image processing,[31,32] object detection,[33] topography [34]; strain analysis,[35,36] and anisotropy analysis,[8,10,11,18,37,38] for example.

Ellipse least-squares fitting methods are widely used and may consist of algebraic or geometric curve fitting. Some authors propose algebraic fitting of conics without constraint,[39,40] arguing that the minimization procedure will converge to an ellipse if this is the best fitting curve. Others prefer ellipse specific fitting adding a constraint to the algebraic equation.[32,41] A third group uses geometric fitting [33,42,43] minimizing the orthogonal distance. In all works cited in this paragraph, cartesian coordinates system was used.

Ellipses can be described by five geometric parameters, which may be the coordinates of the center, both semi-axes and the angle of tilt of one semi-axis. Ellipse fitting can be applied to azimuthal data, which are conveniently represented in polar coordinates. Due to the nature of the data, the ellipse is always centered at the origin of the coordinate system, so the proposed fitting procedure involves determining three curve parameters: both semi-axes and the orientation of one semi-axis referred to a given direction. In this article, a method for fitting ellipses in polar coordinates is proposed.

The equation of an ellipse in polar coordinates is:

$$r' = \frac{ab}{\sqrt{b^2+(a^2-b^2)sin^2(\theta-\theta_0)}} \tag{5}$$

where r' and θ are the polar coordinates of the points on the ellipse, a and b are respectively the major and the minor semi-axes, and θ_0 is the tilt angle of the major semi-axis orientation.

The residual f_i of any i^{th} data point is given by

$$f_i = r_i - r'_i = r_i - \frac{ab}{\sqrt{b^2+(a^2-b^2)sin^2(\theta_i-\theta_0)}} \tag{6}$$

where r_i and θ_i are the coordinates of the ith data point and r'_i is the radius coordinate of the ith estimated point.

A least-squares fitting method was used to define the objective function ϕ to be minimized with respect to a, b, and θ_0:

$$\phi = \sum_{i=1}^{n} f_i^2 = \sum_{i=1}^{n}(r_i^2 - r_i'^2) = \sum_{i=1}^{n}\left[r_i - \frac{ab}{\sqrt{b^2+(a^2-b^2)sin^2(\theta_i-\theta_0)}}\right]^2 \tag{7}$$

where n is the number of data points.

An iterative method without evaluating derivatives was used for the minimization procedure. Due to the small number of curve parameters and of data points typical of the suggested applications, and also to the possibility of a good initial guess for both semi-axes from preliminary data analysis, a simple method for function minimization was chosen consisting of a grid based direct search, as described by Rao.[44] Initial guess adopted for the semi-axes are equal to the maximum and the minimum values of the experimental data used in the fitting procedure, and these can be defined as the center point of the grid. The search range and intervals can be set for each case. The algorithm was implemented using C++ programming language.

The accuracy of the fit was estimated by evaluating the sum of the squared residuals relative to the mean value of the squared data $(r^2)^*$, as described by

$$R_\phi = 1 - \frac{\sum_{i=1}^{n}(r_i^2 - r_i'^2)}{(r^2)^*} \tag{8}$$

The proposed method is intended to be applied to two-dimensional analysis not only of wave velocity but for any directional properties, including those whose principal values may differ by one or more orders of magnitude, as is the case of permeability.

4. Validation tests

Some selected tests are presented in Sections 4.1 and 4.2 to prove the ability of the method to fit ellipses correctly and its robustness.

4.1. Data without noise

Three tests consisted in choosing a set of points on a given ellipse with known semi-axes a and b and tilt angle of the major semi-axis θ_0 to be used as data points to fit an ellipse, which is then compared to the original one. The results of three tests representing different anisotropic materials are shown in Fig. 1. Data set for each test was composed of eight points separated by an angle of 45°. The first curve is a circle, which is representative of an isotropic rock. The second curve is an ellipse with a/b=2, and the third is an ellipse with a/b=10.