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Uniqueness results for a boundary value problem in dipolar thermoelasticity to model composite materials

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ABSTRACT

Composites with microstructure display nonlocal effects and can be effectively modeled through dipolar elasticity. A mixed initial boundary problem is addressed for dipolar thermoelasticity. The basic equations and conditions of the problem are set-up and uniqueness results are proven, which are obtained without introducing definiteness restrictions on the elastic coefficients and, also, without the usual condition imposed to the heat conductivity tensor of being positive definite. An additional result is finally obtained, namely, a reciprocal identity of the Betti's type which underlies another uniqueness result obtained again under weak restrictions.

1. Introduction

Composites are often characterized by hierarchical microstructures providing special characteristics. These microstructures typically introduce nonlocal and gradient effects that can be modeled only through generalized continua [1–4], so that theoretical advances in the treatment of these continua is important for applications in composites.

Elastic bodies with microstructure were initially considered by Eringen (see for instance [5,6]) and developed in Refs. [7–14]. Applications to plates made up of microstretch elastic material were provided by Ref. [11] and Boussinesq-Somigliana-Galerkin solutions were given by Ref. [7] and discussed also in Refs. [10] and [12]. Wave solution for specific bodies with microstructure can be found in Refs. [14] and [15]. Furthermore, the dipolar aspect of the microstructure has been addressed by Mindlin [16], Green and Rivlin [17], Gurtin and Fried [18].

In the present article, a mixed initial boundary value problem is addressed, which is formulated to model a thermoelastic body with dipole structure, and uniqueness results of its solution are provided. Uniqueness of solutions and other related problems have been extensively investigated in the last decade [15–31]. Uniqueness results in elasticity or thermoelasticity are very often based on the assumption that the thermoelastic coefficients or elasticity tensor are positive definite, see Ref. [27]. Other authors approach the same issues by using different types of energy conservation laws. For instance, Green and Laws indicate in Ref. [32] a uniqueness result that is derived by adding certain definiteness assumptions to the restrictions arising from thermodynamics. As an exception to the rule, the uniqueness result of [20] is obtained using weak restrictions. More exactly, Brun employs a specific conservation law of energy and an identity of Lagrange type in order to prove a result of uniqueness in the isothermal case. In the present study the problem of uniqueness is approached of the mixed problem in the case of dipolar bodies by means of some differential identities of the Lagrange type. To be more specific, it will be shown that under weak hypotheses on the thermoelastic coefficients, a solution to the considered problem exists and is unique if the displacement and temperature fields satisfy weak conditions and if the mass density, the specific heat coefficient, and the coefficients of inertia are assumed to be strictly positive.

2. Governing equations

A thermoelastic dipolar composite is assumed to occupy an open domain B of the three-dimensional Euclidian space with closure $\overline{B}$. Assume that the boundary ∂B of B is a closed and bounded surface and refer the deformation of the body to a fixed

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system of rectangular Cartesian axes. Also, the Cartesian tensor notation is adopted, with the usual summation convention on repeated indices. The variable t, $t \in [0, \infty)$, is the time and the spatial variables of points from B are denoted by x_j. The spatial and the time variables of functions will be usually omitted. Greek subscripts are understood to range over the integers $(1, 2)$ and Latin subscripts take the values $1, 2, 3$. A superposed dot denotes the partial differentiation with respect to time, t, and a subscripts preceded by a comma denotes partial differentiation with respect to the corresponding Cartesian coordinate. The theory of thermoelasticity of dipolar bodies as it established in Ref. [17] will be considered and the terminology and notations will be similar to those in Ref. [9] and in Ref. [10].

The basic equations of the mixed problem in this context are:

- the equations of motion

$$\left(\tau_{ij} + \sigma_{ij}\right)_{,j} + F_i = \varrho \ddot{u}_i, \tag{1}$$

$$\mu_{ijk,i} + \sigma_{jk} + G_{jk} = I_{kr} \ddot{\varphi}_{jr}; \tag{2}$$

- the equation of energy:

$$T_0 \dot{\eta} = q_{i,i} + r. \tag{3}$$

The equations (1)–(3) are satisfied for all $(x, t) \in B \times [0, \infty)$.

The stress tensors of components $\tau_{ij}, \sigma_{ij}, \mu_{ijk}$, the heat conduction vector of components q_i and the specific entropy η are defined by means of the constitutive equations

$$\begin{aligned}
&\tau_{ij} = C_{ijmn}\varepsilon_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr} - D_{ij}\vartheta, \\
&\sigma_{ij} = G_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} - E_{ij}\vartheta, \\
&\mu_{ijk} = F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + A_{ijkmnr}\chi_{mnr} - F_{ijk}\vartheta, \\
&\varrho\eta = a\vartheta + D_{ij}\varepsilon_{ij} + E_{ij}\gamma_{ij} + F_{ijk}\chi_{ijk}, \\
&q_i = k_{ij}\vartheta_{,j}.
\end{aligned} \tag{4}$$

satisfied for all $(x, t) \in B \times [0, \infty)$.

To insert the components of strain tensors, $\varepsilon_{ij}, \gamma_{ij}, \chi_{ijk}$, we will use the following geometric equations

$$2\varepsilon_{ij} = u_{j,i} + u_{i,j}, \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \quad \chi_{ijk} = \varphi_{ij,k}. \tag{5}$$

Other notations that we have used in previous equations, mean ϱ-the constant density which is assumed be strictly positive, T_0-the constant temperature, ϑ-the temperature measured from the constant temperature T_0, F_i-the body force per unit mass, G_{jk}-the body couple per unit mass, r-the heat supply per unit mass and unit time, I_{ij}-the micro-inertia which is positive definite, $C_{ijmn}, B_{ijmn}, \ldots, a$-the characteristic constants of material, where a is assumed be strictly positive or strictly negative. The previous elastic coefficients satisfy following symmetry relations

$$\begin{aligned}
C_{mnij} &= C_{ijmn} = C_{ijnm}, \\
B_{ijmn} &= B_{mnij}, \\
G_{ijmn} &= G_{ijnm}, \\
F_{ijkmn} &= F_{ijknm}, \\
A_{ijkmnr} &= A_{mnrijk}, \\
E_{ij} &= E_{ji}, \\
k_{ij} &= k_{ji}.
\end{aligned} \tag{6}$$

The entropy inequality implies

$$k_{ij}\vartheta_i\vartheta_j \geq 0. \tag{7}$$

To complete the equations (1)–(5) we will consider the prescribed initial conditions that follow

$$\begin{aligned}
u_i(x_s, 0) &= a_i(x_s), \\
\dot{u}_i(x_s, 0) &= b_i(x_s), \\
\varphi_{jk}(x_s, 0) &= c_{jk}(x_s), \\
\dot{\varphi}_{jk}(x_s, 0) &= d_{jk}(x_s), \\
\eta(x_s, 0) &= \eta_0(x_s),
\end{aligned} \tag{8}$$

which are satisfied for all $(x_k) \in \overline{B}$. Also, if we consider the subsets ∂B_1, ∂B_2 and ∂B_3 of the surface ∂B which, together with their complements ∂B_1^c, ∂B_2^c and ∂B_3^c, satisfy the following conditions

$$\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B,$$

$$\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \emptyset,$$

then we can add the following boundary conditions

$$\begin{aligned}
&u_i = \tilde{u}_i \text{ on } \partial B_1 \times [0, \infty), \quad t_i = \tilde{t}_i \text{ on } \partial B_1^c \times \left[0, \infty\right) \\
&\varphi_{jk} = \tilde{\varphi}_{jk} \text{ on } \partial B_2 \times [0, \infty), \quad m_{jk} = \tilde{m}_{jk} \text{ on } \partial B_2^c \times \left[0, \infty\right) \\
&\vartheta = \tilde{\vartheta} \text{ on } \partial B_3 \times [0, \infty), \quad q = \tilde{q} \text{ on } \partial B_3^c \times [0, \infty)
\end{aligned} \tag{9}$$

where t_i, m_{jk} and q are defined by

$$t_i = (\tau_{ij} + \sigma_{ij})n_j, m_{jk} = \mu_{ijk}n_i, \ q = q_i n_i$$

and represent the components of surface traction, the components of surface couple and the heat flux, respectively.

In (8) a_i, b_i, c_{jk}, d_{jk} and η_0 are prescribed functions. Also, in (9) $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_{jk}$, $\tilde{m}_{jk}$, $\tilde{\vartheta}$ and $\tilde{q}$ are given functions.

Introducing (4) and (5) in (1)–(3), we obtain the following system

$$\begin{aligned}
\varrho\ddot{u}_i \ &= \big[(C_{ijmn} + G_{ijmn})u_{n,m} + (G_{mnij} + B_{ijmn})(u_{n,m} - \varphi_{mn}) + \\
&\quad + (F_{mnrij} + D_{ijmnr})\varphi_{nr,m} - (D_{ij} + E_{ij})\vartheta\big]_{,j} + \varrho F_i, \\
I_{kr}\ddot{\varphi}_{jr} \ &= \Big[F_{ijkmn}u_{n,m} + D_{mnijk}(u_{n,m} - \varphi_{mn}) + A_{ijkmnr}\varphi_{nr,m} - F_{ijk}\vartheta\Big]_{,i} + \\
&\quad + G_{jkmn}u_{m,n} + B_{jkmn}(u_{n,m} - \varphi_{mn}) + D_{jkmnr}\varphi_{nr,m} - E_{jk}\vartheta\Big] + \varrho G_{jk}, \\
aT_0\dot{\vartheta} &= -T_0\Big[D_{ij}v_{j,i} + E_{ij}\left(v_{j,\,i} - \psi_{ij}\right) + F_{ijk}\psi_{jk,i}\Big] + \left(k_{ij}\vartheta_{,j}\right)_{,i} + \varrho r,
\end{aligned} \tag{10}$$