Unified matrix-exponential FDTD formulations for modeling electrically and magnetically dispersive materials

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A B S T R A C T

Unified matrix-exponential finite difference time domain (ME-FDTD) formulations are presented for modeling linear multi-term electrically and magnetically dispersive materials. In the proposed formulations, Maxwell’s curl equations and the related dispersive constitutive relations are cast into a set of first-order differential matrix system and the field’s update equations can be extracted directly from the matrix-exponential approximation. The formulations have the advantage of simplicity as it allows modeling different linear dispersive materials in a systematic manner and also can be easily incorporated with the perfectly matched layer (PML) absorbing boundary conditions (ABCs) to model open region problems. Apart from its simplicity, it has been shown that the proposed formulations necessitate less storage requirements as compared with the well-know auxiliary differential equation FDTD (ADE-FDTD) scheme while maintaining the same accuracy performance.

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1. Introduction

The finite difference time domain (FDTD) method [1] has been found to be one of the most popular tools in computational electrodynamics. In [2–5], different extensions of this method have been successfully introduced for modeling linear Debye, Drude or Lorentz electrically dispersive materials. These formulations can be categorized into three categories: 1) the recursive convolution (RC) method [2] and its improved piecewise linear RC (PLRC) approach [3], 2) the auxiliary differential equation (ADE) method [4], and 3) the $Z$-transform method [5]. Very recently, another dispersive FDTD formulations, based on the matrix-exponential approach, have been introduced in [6,7]. Nevertheless, these formulations stand for linear single-term Debye [6] or Drude [7] electrically dispersive materials and they need to be reformulated each time the material’s frequency dependent model is changed.

In this paper, unified matrix-exponential dispersive FDTD formulations are presented for modeling multi-term linear electrically and magnetically dispersive materials. In the proposed formulations, Maxwell’s curl equations and the related dispersive constitutive relations are cast into a set of first-order differential matrix system and the field’s update equations can be extracted directly from the matrix-exponential approximation. The formulations have the advantage of simplicity as it allows modeling different linear multi-term electrically and magnetically dispersive materials in a systematic manner and also can be easily incorporated with the perfectly matched layer (PML) absorbing boundary conditions (ABCs) [8] to model open region problems. Apart from its simplicity, it has been shown that the proposed formulations necessitate less storage requirements as compared with the well-know auxiliary differential equations FDTD (ADE-FDTD) [4] approach while maintaining the same accuracy performance.

The paper is organized in the following manner. In Section 2, the formulations of the proposed approach are presented. In Section 3, the results of a numerical test are included to show the validity of the proposed formulations, and finally, summary and conclusions are included in Section 4.

2. Formulations

Considering a lossy and linear isotropic dispersive domain, Maxwell’s curl equations can be written as

$$\begin{align*}
\left( j\omega \varepsilon_0 \sigma_r + \sigma_0 \right) \overline{\mathbf{E}} &= \mathbf{C} \overline{\mathbf{H}} \\
\left( j\omega \mu_0 \mu_r + \sigma_0 \right) \overline{\mathbf{H}} &= \mathbf{C}^T \overline{\mathbf{E}}
\end{align*}$$

where $\overline{\mathbf{E}} = [\overline{E_x}, \overline{E_y}, \overline{E_z}]^T$, $\overline{\mathbf{H}} = [\overline{H_x}, \overline{H_y}, \overline{H_z}]^T$, $\sigma_r$ and $\sigma_0$ are, respectively, the electric and the magnetic conductivities which are assumed to be constant in this paper, $\mathbf{C}$ is the matrix curl operator defined as

$$\mathbf{C} = \left( \begin{array}{ccc} 0 & -\partial_z & -\partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{array} \right)$$

and $\varepsilon_r(\omega)$ and $\mu_r(\omega)$ are, respectively, the electric permittivity and the magnetic permeability, which can be written for a two-term Lorentz model, for example, as

$$\varepsilon_r(\omega) = \varepsilon_\infty \left( 1 + \frac{\omega_p^2}{\omega^2 + \omega_0^2} \right)$$

and

$$\mu_r(\omega) = \mu_\infty \left( 1 + \frac{\omega_p^2}{\omega^2 + \omega_0^2} \right)$$
\( \varepsilon_r(\omega) = \varepsilon_\infty + \frac{2}{\omega_c} \sum_{k=1}^{2} \frac{G_{ek}(\varepsilon_k - \varepsilon_\infty)\omega_p^2}{(j\omega)^2 + j\omega\Gamma_{ek} + \omega_c^2} \) \hfill (4)

\( \mu_r(\omega) = \mu_\infty + \frac{2}{\omega_h} \sum_{k=1}^{2} \frac{G_{hk}(\mu_k - \mu_\infty)\omega_p^2}{(j\omega)^2 + j\omega\Gamma_{hk} + \omega_h^2} \) \hfill (5)

with \( \varepsilon_k = \varepsilon_k(0), \varepsilon_\infty = \varepsilon_\infty(\infty), \mu_k = \mu_k(0), \mu_\infty = \mu_\infty(\infty), G_{ek} \) and \( G_{hk} \) are, respectively, the polarizability and core relaxation time of the medium, \( \omega_p \) is the plasma frequency, \( \omega_c \) is the resonance frequency, and \( \Gamma_{ek} \) is the damping factor. Substituting (4) and (5) into (1) and (2) and introducing the following auxiliary variables:

\[ \tilde{J}_k = j\omega \omega_p \sqrt{G_{ek}(\varepsilon_k - \varepsilon_\infty)/\varepsilon_\infty} \tilde{E}_k \] \hfill (6)

\[ \tilde{M}_k = j\omega \omega_h \sqrt{G_{hk}(\mu_k - \mu_\infty)/\mu_\infty} \tilde{H}_k \] \hfill (7)

where \( \tilde{J}_k = [J_{y1}, J_{y2}, J_{y3}]^T \), \( \tilde{P}_k = [\tilde{P}_{y1}, \tilde{P}_{y2}, \tilde{P}_{y3}]^T \), \( \tilde{M}_k = [\tilde{M}_{y1}, \tilde{M}_{y2}, \tilde{M}_{y3}]^T \), and \( \tilde{K}_k = [\tilde{K}_{y1}, \tilde{K}_{y2}, \tilde{K}_{y3}]^T \), then (1), (2), and (6)-(9) can be written in the time domain in a matrix form as

\[ \frac{\partial \omega}{\partial t} = \mathfrak{R} \] \hfill (10)

where \( \omega = [E, H, J_1, J_2, J_3, P_1, P_2, M_1, M_2, M_3]^T \),

\[ \mathfrak{R} = \begin{bmatrix} \alpha_{X_{y1}} & \alpha_{X_{y2}} & \alpha_{X_{y3}} & \alpha_{X_{y2}} \\ X_{y1} & 0 & 0 & 0 \\ 0 & Y_{y1} & 0 & 0 \\ 0 & 0 & Y_{y1} & 0 \\ 0 & 0 & 0 & Y_{y2} \end{bmatrix} \quad \mathfrak{M} = \begin{bmatrix} \tilde{J}_{y3} & \nu \tilde{C}_{y3} \\ \tilde{J}_{y3} & 0 \\ \tilde{J}_{y3} & 0 \\ \tilde{J}_{y3} & 0 \\ \tilde{J}_{y3} & 0 \end{bmatrix} \] \hfill (11)

\[ A_{ek} = \begin{bmatrix} A_{ek} & 0 & 0 & 0 \\ 0 & A_{ek} & 0 & 0 \\ 0 & 0 & A_{ek} & 0 \\ 0 & 0 & 0 & A_{ek} \end{bmatrix} \quad B_{ek} = \begin{bmatrix} B_{ek} \\ 0 \\ 0 \\ 0 \end{bmatrix} \] \hfill (12)

where \( A_{ek} \) and \( B_{ek} \) are, respectively, the \( n \times n \) null and the identity matrices, \( \nu = 1/\sqrt{\varepsilon_\infty \mu_\infty} \) is the speed of light in the dispersive medium, \( \alpha = -1, \quad \tilde{\sigma}_u = \frac{\sigma_u}{D_{uu}D_{uu}} \) \hfill (13)

and \( \Delta t = \frac{\Delta t}{\Delta t} \) for \( u \) or \( h \)

(14)

with \( D_{uu} = \varepsilon_\infty \nu \) and \( D_{uu} = \mu_\infty \nu \) for \( u \) or \( h \).

\[ \tilde{R}_h = \begin{bmatrix} \tilde{R}_{x1} & \tilde{R}_{x2} & \tilde{R}_{x3} \\ \tilde{R}_{x1} & 0 & 0 \\ \tilde{R}_{x1} & 0 & 0 \\ \tilde{R}_{x1} & 0 & 0 \end{bmatrix} \] \hfill (15)

\[ \mathfrak{M} = \begin{bmatrix} \tilde{M}_{y1} & \tilde{M}_{y2} & \tilde{M}_{y3} \\ \tilde{M}_{y1} & 0 & 0 \\ \tilde{M}_{y1} & 0 & 0 \\ \tilde{M}_{y1} & 0 & 0 \end{bmatrix} \]

\[ \mathfrak{A}_h = \begin{bmatrix} \mathfrak{A}_{x1} & \mathfrak{A}_{x2} & \mathfrak{A}_{x3} & \mathfrak{A}_{x2} \\ \mathfrak{A}_{x1} & 0 & 0 & 0 \\ \mathfrak{A}_{x1} & 0 & 0 & 0 \\ \mathfrak{A}_{x1} & 0 & 0 & 0 \end{bmatrix} \] \hfill (16)

Using (18)-(20), the following can be obtained from (25) and (26), respectively.