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An efficient, high-order finite element method using the nodal averaging technique for incompressible fluid flows

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ABSTRACT

A new finite element method is presented for use of quadrilateral nine-node elements in the solution of the incompressible Navier–Stokes equations. In a conventional predictor–corrector scheme, the method applies the nodal averaging technique to discretize the Poisson equation used for the simultaneous relaxation of velocity and pressure. Additionally, efficient approximation procedures are devised to increase the speed of computation without deteriorating solution accuracy. The proposed numerical schemes are evaluated on two-dimensional test problems including a classical lid-driven cavity flow and a flow over a backward-facing step in a flow channel. The results show good accuracy even when distorted elements are used for calculation.

1. Introduction

Finite element methods for fluid flows have been extensively studied over the last four decades and certain computational schemes are now widely used in research and commercial numerical codes (see Refs. [1–3] and the many references therein). There are of course many other numerical techniques that are capable of solving various challenging flow problems with accuracy (for example, see Refs. [4–6] and the references therein). The research effort to develop efficient, robust, and accurate numerical methods is still required together with the increase in performance of supercomputers.

For the numerical analyses of incompressible fluid flows, the segregated methods, which are evolved from the marker-and-cell (MAC) method [7], are commonly used to discretize the Navier–Stokes equations. In the finite difference discretization, the simplified MAC (SMAC) method was developed by applying the predictor–corrector method to the momentum equation in which the pressure is implicitly discretized in time [8], and then the highly simplified MAC (HSMAC) method was proposed as a more efficient approach, which was developed by adding the simultaneous relaxation scheme for modifying velocity and pressure to the SMAC method [9]. Similar methods, some of which are known as the semi-implicit method for pressure-linked equations (SIMPLE) [10] and its modified variant, SIMPLE Revised (SIMPLER) [11], were also devised within the framework of the predictor–corrector procedure; in those methods, the calculation approach for solving the pressure is different from that for the SMAC method. An extended method based on the HSMAC algorithm was proposed by Tanahashi et al.; they converted the HSMAC method into the one for the finite element discretization together with an ingenuity at solving high Reynolds number fluid flows [12]. This method was named the generalized simplified MAC (GSMAC) method and has been applied to various flow problems, e.g., electrically conducting fluid flow [13], free-surface flow [14,15], and visco-elastic flow [16]. While much research effort has been expended with this finite element technique, linear (bilinear or trilinear) shape functions have been used for the interpolation of velocity and the accuracy of the scheme has almost always been examined with undistorted grids (i.e., rectangular and rectangular parallelepiped grids). An improved approach that employs higher-order interpolation functions for both velocity and pressure is desirable, and the method needs to be robust even if it is applied to a distorted mesh.

The objective in this paper is to present a new finite element technique using nine-node elements in an efficient way for two-dimensional (2D) solutions of incompressible fluid flow problems. An important feature is that the nodal averaging technique is incorporated into the simultaneous relaxation procedure in order to easily obtain the amount of correction for pressure at the grid nodes. In fact, this idea can also be applied to non-quadrilateral elements. In addition, two approximation methods are invented for the calculation of the coefficient matrices obtained in the finite element discretization of the Navier–Stokes equations. In these methods, the integrands are approximated so

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as to divide the coefficient matrices into element-dependent and element-independent parts, which leads to significant speed-up of computation compared with the case where a conventional numerical integration method (e.g., Gaussian quadrature rule) is used for the calculation of the integrals.

The paper is organized as follows. In Section 2, the governing equations and the finite element discretization procedures are fully described together with the approximation schemes described above. Section 3 presents demonstrative numerical test and solutions to illustrate the capability of the proposed methods. The solutions include the use of undistorted and distorted elements, with coarse and fine meshes, in the analysis of a well-known lid-driven cavity flow and a flow in a backward-facing step geometry. Lastly, conclusions of the present work are given in Section 4.

2. Numerical method

2.1. Governing equations and time marching algorithm

The governing equations are the equation of continuity and the momentum equation for incompressible fluid flow, which are, respectively, written in dimensionless form as follows:

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{v}, \tag{2}$$

where $\boldsymbol{v}$ is the velocity, t is the time, p is the pressure, and Re is the Reynolds number. Here, the external force is not taken into account in Eq. (2). Since pressure waves propagate through incompressible fluid instantly, the pressure in the momentum equation needs to be discretized implicitly in time while the velocity can be discretized explicitly. As a result, Eq. (2) is discretized in time in the following form:

$$\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta t} = -\left(\boldsymbol{v}^n \cdot \nabla\right) \boldsymbol{v}^n - \nabla p^{n+1} + \frac{1}{Re}\nabla^2 \boldsymbol{v}^n, \tag{3}$$

where the superscript n denotes the n-th time step, and Δt is the increment of time.

According to the fractional step approach [9,12], Eq. (3) is divided into the following two equations:

$$\frac{\widetilde{\boldsymbol{v}} - \boldsymbol{v}^n}{\Delta t} = -\left(\boldsymbol{v}^n \cdot \nabla\right) \boldsymbol{v}^n - \nabla p^n + \frac{1}{Re}\nabla^2 \boldsymbol{v}^n, \tag{4}$$

$$\frac{\boldsymbol{v}^{n+1} - \hat{\boldsymbol{v}}^{(m)}}{\Delta t} = -\nabla\left(p^{n+1} - \hat{p}^{(m)}\right). \tag{5}$$

Here, $\widetilde{\boldsymbol{v}}$ in Eq. (4) is the intermediate value of velocity, which is explicitly obtained from this equation as a predictor step. The velocity $\hat{\boldsymbol{v}}^{(m)}$ and pressure $\hat{p}^{(m)}$ in Eq. (5) correspond to the solution of the following Poisson equation at the m-th iteration step:

$$\nabla^2 \phi_p = \nabla \cdot \hat{\boldsymbol{v}}, \tag{6}$$

where

$$\phi_p \equiv \left(p^{n+1} - \hat{p}\right) \Delta t. \tag{7}$$

Eq. (6) is obtained from the divergence of Eq. (5) and the requirement that $\nabla \cdot \boldsymbol{v}^{n+1} = 0$. The velocity $\hat{\boldsymbol{v}}$ and pressure $\hat{p}$ are iteratively calculated through a Newton–Raphson method with the initial guess $\hat{\boldsymbol{v}}^{(0)} = \widetilde{\boldsymbol{v}}$ and $\hat{p}^{(0)} = p^n$ until the value of $\nabla \cdot \hat{\boldsymbol{v}}$ satisfies the convergence criterion. This is called the simultaneous relaxation method, and its detailed calculation method for the present scheme will be described in Section 2.3. For the moment, let $\delta\phi_p^{(m)}$ be the amount of correction for the modified potential $\phi_p^{(m)}$ at the m-th iteration step, i.e., $\delta\phi_p^{(m)} = \phi_p^{(m+1)} - \phi_p^{(m)}$. Then, the velocity $\hat{\boldsymbol{v}}$ and pressure $\hat{p}$ are corrected as follows:

$$\hat{\boldsymbol{v}}^{(m+1)} = \hat{\boldsymbol{v}}^{(m)} + \nabla \delta\phi_p^{(m)}, \tag{8}$$

$$\hat{p}^{(m+1)} = \hat{p}^{(m)} - \frac{\delta\phi_p^{(m)}}{\Delta t}, \tag{9}$$

where Eqs. (8) and (9) are derived from Eqs. (5) and (7), respectively. After the error of the equation of continuity becomes negligibly small, the iteratively corrected velocity and pressure values are replaced by $\boldsymbol{v}^{n+1} = \hat{\boldsymbol{v}}^{(m+1)}$ and $p^{n+1} = \hat{p}^{(m+1)}$, respectively.

2.2. Finite element formulation for the predictor step

Next, the equations obtained in Section 2.1 are further discretized in space based on a finite element procedure. Here, the velocity components are interpolated using piecewise quadratic shape functions, while the pressure is interpolated using piecewise linear shape functions due to the requirement known as the inf–sup condition [17,18]. First, with the use of a Galerkin method, Eq. (4), the predictor step, is spatially discretized as follows:

$$\overline{M}_{ij}\frac{\widetilde{\boldsymbol{v}}_j - \boldsymbol{v}_j^n}{\Delta t} = -A_{ij}^n \boldsymbol{v}_j^n + \boldsymbol{C}_{ik} p_k^n - \frac{1}{Re} D_{ij} \boldsymbol{v}_j^n - \boldsymbol{S}_i, \tag{10}$$

where the subscripts i and j denote the global node number for velocity, while the subscript k denotes the global node number for pressure. Here, the summation convention applies to the subscripts j and k. The last term on the right-hand side of Eq. (10), $\boldsymbol{S}_i$, corresponds to the boundary integral expression in which non-Dirichlet boundary condition(s) is (are) incorporated. It is assumed here that the boundary integral term $\boldsymbol{S}_i$ does not depend on time. The coefficient matrices in Eq. (10) are given by the following:

$$\begin{aligned}
M_{ij} &= \int_\Omega N_i^{(2)} N_j^{(2)} \mathrm{d}\Omega, \\
A_{ij}^n &= \int_\Omega \boldsymbol{v}^n \cdot \left(N_i^{(2)} \nabla N_j^{(2)}\right) \mathrm{d}\Omega, \\
\boldsymbol{C}_{ik} &= \int_\Omega \left(\nabla N_i^{(2)}\right) N_k^{(1)} \mathrm{d}\Omega, \\
D_{ij} &= \int_\Omega \nabla N_i^{(2)} \cdot \nabla N_j^{(2)} \mathrm{d}\Omega, \\
\boldsymbol{S}_i &= \int_{\Gamma_{\mathbb{I}}} N_i^{(2)} \left(p\boldsymbol{n} - \frac{1}{Re}\frac{\partial \boldsymbol{v}}{\partial n}\right) \mathrm{d}\Gamma,
\end{aligned} \tag{11}$$

where Ω represents the calculation domain and $\Gamma_{\mathbb{I}}$ is its boundary where the velocity is not prescribed, $N_i^{(2)}$ (or $N_j^{(2)}$) and $N_k^{(1)}$ are the piecewise quadratic and linear shape functions, respectively, and $\boldsymbol{n}$ is the outward-pointing unit normal vector on the boundary. Strictly speaking, M_{ij}, A_{ij}^n, $\boldsymbol{C}_{ik}$, and D_{ij} in Eq. (11) show the coefficient matrix elements for some specific global nodes i and j in scalar or vector form; however, they are simply called the coefficient matrices in this paper. Note that the mass matrix M_{ij} is approximated by the lumped mass matrix $\overline{M}_{ij}$ [19] in Eq. (10), which is defined by

$$\overline{M}_{ij} = \delta_{ij} \int_\Omega N_i^{(2)} \mathrm{d}\Omega, \tag{12}$$

where δ_{ij} is the Kronecker delta.

2.3. Simultaneous relaxation method for velocity and pressure

Since the intermediate value of velocity, $\widetilde{\boldsymbol{v}}$, at the predictor step does not satisfy the equation of continuity in calculation space in general, the velocity needs to be corrected through