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Holes, cracks, or inclusions in two-dimensional linear anisotropic viscoelastic solids

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A B S T R A C T

By combining the elastic-viscoelastic correspondence principle with the analytical solutions of anisotropic elasticity, the problems of two-dimensional linear anisotropic viscoelastic solids can be solved directly in the Laplace domain. After getting the solutions in the Laplace domain, their associated solutions in real time domain can be determined by numerical inversion of Laplace transform. Following this general adopted process, the problems of holes, cracks, or inclusions in two-dimensional linear anisotropic viscoelastic solids, which appear frequently in polymer matrix composites and cannot be solved directly by the commonly used commercial finite elements, are solved in this paper. Here, the hole can be elliptical or polygon-like; the crack can be a single crack, or two collinear cracks, or an interface crack; and the inclusion can be rigid, elastic or viscoelastic. The loads considered include the uniform load at infinity, and the point force applied at the arbitrary location. The solution of the point force is then employed as the fundamental solution of boundary element method which is used for further comparison of the analytical solutions. The accuracy and efficiency of the presented solutions are illustrated through four representative numerical examples which involve four isotropic viscoelastic and two anisotropic viscoelastic materials.

1. Introduction

Polymer matrix composites exhibit not only anisotropic (directional-dependent) but also viscoelastic (time-dependent) behaviors. Although there are many different kinds of commercial software working on the stress analysis of composite materials, most of them only provide the functions for isotropic elastic, anisotropic elastic, or isotropic viscoelastic materials, almost none of them consider the analysis of anisotropic viscoelastic solids. Additional works are required for some of them.

The elastic-viscoelastic correspondence principle was proposed long time ago [1]. It states that a problem in linear elasticity is identical to one in viscoelasticity in the transformed domain. Although this principle is simple and is applicable for a time-independent boundary value problem, due to the complexity of the analytical solutions for the corresponding problems of anisotropic elasticity, its application to the anisotropic viscoelastic solids is not that direct. And hence, not too many results have been presented for the *anisotropic viscoelastic* solids by combining the correspondence principle with the analytical solutions of anisotropic elasticity. Most of the successful applications are restricted to the cases of isotropic viscoelastic solids – homogeneous or nonhomogeneous, such as [2–6], etc. For the cases of anisotropic viscoelastic solids, only the application to finite element method (FEM), boundary element method (BEM) and the problems of interface corners have been discussed [7–9].

Since several analytical solutions have been presented for the problems with holes/cracks/inclusions in anisotropic elastic solids [10], in this paper we try to provide new results of their corresponding anisotropic viscoelastic solids by using the correspondence principle. Through the use of this principle, the well-known Stroh complex variable formalism in the Laplace domain of viscoelasticity can be proved to have the same mathematical form as that of anisotropic elasticity [9]. After getting the displacements, strains and stresses in the Laplace transform domain from the solutions obtained by Stroh formalism, their associated solutions in real time domain can be determined by numerical inversion of Laplace transform. In this paper Schapery method is adopted to transfer a series of data in Laplace domain into time domain [11]. To show the correctness of our semi-analytical solutions, several examples are illustrated with comparison made by finite element method and boundary element method. These examples include the two-

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dimensional anisotropic viscoelastic solids containing (1) an elliptical or polygon-like hole, (2) a single crack or two collinear cracks, (3) an interface crack, and (4) an elliptical or polygon-like inclusion.

2. Linear anisotropic viscoelasticity

In a fixed Cartesian coordinate system $x_i, i = 1,2,3$, let u_i, σ_{ij}, and ε_{ij} be, respectively, the displacement, stress and strain. The basic equations of linear anisotropic viscoelasticity can be written as [12].

$$\begin{aligned}&\sigma_{ij}(t) = C_{ijkl}(t)\varepsilon_{kl}(0) + \int_0^t C_{ijkl}(t-\tau)\frac{\partial\varepsilon_{kl}(\tau)}{\partial\tau}d\tau,\\ &\varepsilon_{ij}(t) = \frac{1}{2}\{u_{i,j}(t) + u_{j,i}(t)\},\\ &\sigma_{ij,j}(t) = 0,\quad i,j,k,l = 1,2,3,\end{aligned} \tag{2.1}$$

in which the repeated indices imply summation, the subscript comma stands for differentiation, and $C_{ijkl}(t)$ is the elastic tensor (also known to as *relaxation function*) which is assumed to be fully symmetric. To obtain the elastic tensor, one may perform the relaxation test, in which the material is subjected to a sudden strain that is kept constant over the test duration, and the stress is measured over time. Alternatively, one may perform the creep test to obtain the compliance tensor (also known as *creep function*), and then calculate the elastic tensor through inversion.

Consider two-dimensional deformation and follow the derivation procedure of Stroh formalism for linear anisotropic elasticity, it has been proved that the general solution satisfying all the basic equation (2.1) of linear anisotropic viscoelasticity can be expressed in matrix form as [9].

$$\begin{aligned}&\mathbf{u}(\mathbf{x},t) = 2\mathrm{Re}\{\mathbf{A}(t) * \mathrm{d}\mathbf{f}(z,t)\},\\ &\boldsymbol{\phi}(\mathbf{x},t) = 2\mathrm{Re}\{\mathbf{B}(t) * \mathrm{d}\mathbf{f}(z,t)\},\end{aligned} \tag{2.2a}$$

where

$$\begin{aligned}&\mathbf{u} = \begin{Bmatrix} u_1\\ u_2\\ u_3\end{Bmatrix},\ \boldsymbol{\phi} = \begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{Bmatrix},\ \mathbf{f}(z,t) = \begin{Bmatrix} f_1(z_1,t)\\ f_2(z_2,t)\\ f_3(z_3,t)\end{Bmatrix},\\ &\mathbf{A}(t) = [\mathbf{a}_1(t)\quad \mathbf{a}_2(t)\quad \mathbf{a}_3(t)],\\ &\mathbf{B}(t) = [\mathbf{b}_1(t)\quad \mathbf{b}_2(t)\quad \mathbf{b}_3(t)],\\ &z_k = x_1 + \mu_k x_2,\ k = 1,2,3.\end{aligned} \tag{2.2b}$$

In (2.2b), Re stands for the real part, $\phi_i,\ i = 1,2,3$ are the stress functions related to the stresses by

$$\sigma_{i1} = -\phi_{i,2},\ \sigma_{i2} = \phi_{i,1}. \tag{2.3}$$

$\mu_k,\ k = 1,2,3$ are the material eigenvalues which have been proved to be complex and independent of time for standard linear viscoelastic solids, and $\mathbf{a}_k(t)$ and $\mathbf{b}_k(t)$ are their associated eigenvectors. $f_k(z_k,t),\ k = 1,2,3$ are holomorphic complex functions with variables z_k and t. The operator * denotes the Stieltjes convolution, e.g.,

$$\begin{aligned}\mathbf{A}(t)*\mathrm{d}\mathbf{f}(z,t) &= \int_{-\infty}^{t}\mathbf{A}(t-\tau)\mathrm{d}\mathbf{f}(z,\tau)\\ &= \mathbf{A}(t)\mathbf{f}(z,0) + \int_0^t \mathbf{A}(t-\tau)\frac{\partial\mathbf{f}(z,\tau)}{\partial\tau}\mathrm{d}\tau,\end{aligned} \tag{2.4}$$

where the second equality is obtained under the condition that $\mathbf{f}(z,t) = 0$ when $t < 0$.

3. Correspondence principle

Taking the Laplace transform of (2.1) yields

$$\begin{aligned}&\breve{\sigma}_{ij}(s) = s\breve{C}_{ijkl}(s)\breve{\varepsilon}_{kl}(s), \breve{\varepsilon}_{ij}(s) = \frac{1}{2}\left\{\breve{u}_{i,j}(s) + \breve{u}_{j,i}(s)\right\},\\ &\breve{\sigma}_{ij,j}(s) = 0,\end{aligned} \tag{3.1}$$

in which the over-breve, $\breve{\bullet}$, denotes the Laplace transform defined by

$$\breve{f}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t \equiv L\{f(t)\}. \tag{3.2}$$

Since (3.1) are identical to the basic equations of linear anisotropic elasticity, if the boundary of a viscoelastic body is invariant with time, the viscoelastic solutions in the Laplace domain can be obtained directly from the solutions of the corresponding elastic problems by replacing the elastic stiffness tensor C_{ijkl} with $s\breve{C}_{ijkl}(s)$. This statement is the so-called *elastic-viscoelastic correspondence principle* [2], [13–17].

By applying the correspondence principle and Stroh formalism for two-dimensional linear anisotropic elasticity [10,18], the general solutions satisfying the 15 partial differential equation (3.1) can be written as

$$\breve{\mathbf{u}}(\mathbf{x},s) = 2\mathrm{Re}\{\mathbf{A}_s(s)\mathbf{f}_s(z,s)\}, \breve{\boldsymbol{\phi}}(\mathbf{x},s) = 2\mathrm{Re}\{\mathbf{B}_s(s)\mathbf{f}_s(z,s)\}, \tag{3.3a}$$

where

$$\begin{aligned}&\breve{\mathbf{u}} = \begin{Bmatrix}\breve{u}_1\\ \breve{u}_2\\ \breve{u}_3\end{Bmatrix},\ \breve{\boldsymbol{\phi}} = \begin{Bmatrix}\breve{\phi}_1\\ \breve{\phi}_2\\ \breve{\phi}_3\end{Bmatrix},\ \mathbf{f}_s(z,s) = \begin{Bmatrix} f_1^s(z_1,s)\\ f_2^s(z_2,s)\\ f_3^s(z_3,s)\end{Bmatrix},\\ &\mathbf{A}_s(s) = [\mathbf{a}_1^s(s)\quad \mathbf{a}_2^s(s)\quad \mathbf{a}_3^s(s)],\\ &\mathbf{B}_s(s) = [\mathbf{b}_1^s(s)\quad \mathbf{b}_2^s(s)\quad \mathbf{b}_3^s(s)],\\ &z_k = x_1 + \mu_k^s x_2,\quad k = 1,2,3.\end{aligned} \tag{3.3b}$$

$\breve{\mathbf{u}}$ and $\breve{\boldsymbol{\phi}}$ are, respectively, the vectors of displacements and stress functions in the Laplace domain. $f_k^s(z_k,s),\ k = 1,2,3$ are holomorphic complex functions with variables z_k and s. μ_α^s and $(\mathbf{a}_\alpha^s, \mathbf{b}_\alpha^s)$ are the material eigenvalues and eigenvectors in the Laplace domain.

By taking the Laplace transform of (2.2a) and comparing the results with (3.3a), we get

$$s\breve{\mu}_k = \mu_k^s,\quad \breve{\mathbf{A}} = \mathbf{A}_s,\quad \breve{\mathbf{B}} = \mathbf{B}_s,\quad s\breve{\mathbf{f}} = \mathbf{f}_s. \tag{3.4}$$

4. Analytical solutions for problems with holes/cracks/inclusions

From the correspondence principle stated in the previous section, we know that even no analytical solution has been presented for the problems with holes, cracks and inclusions in anisotropic viscoelastic materials they can still be solved from their corresponding anisotropic elastic problems if the boundary of a viscoelastic solid is invariant with time. Since the Stroh formalism in Laplace domain of viscoelasticity is exactly the same as that of anisotropic elastic materials, the solutions of the complex function vector $\mathbf{f}_s(z,s)$ in the Laplace domain will be exactly the same as those of anisotropic elastic problems. With this understanding, for